A strict epistemic approach to physics

Part 1

Background

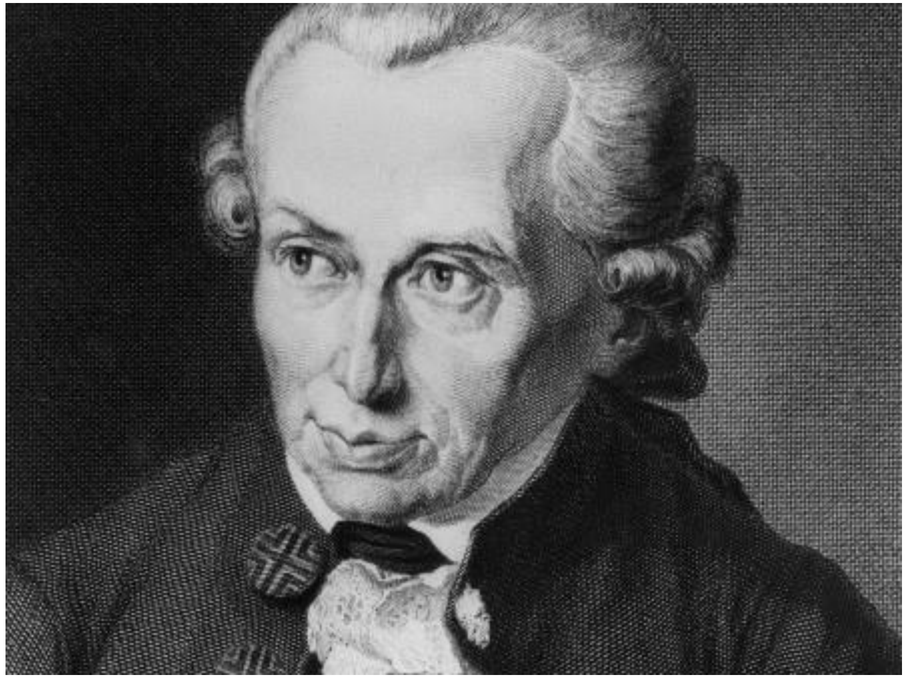
The basic picture

The measurement process

Probability

Elements of quantum mechanics

Kant’s Copernican revolution



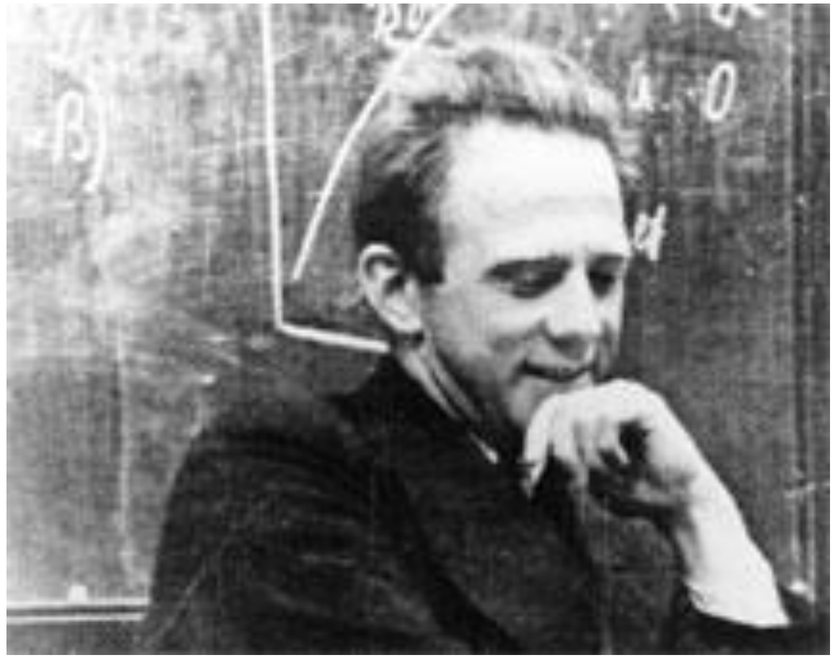
“Up to now it has been assumed that all our cognition must conform to the objects; but all attempts to find out something about them *a priori* through concepts that would extend our cognition have, on this presupposition, come to nothing. Hence let us once try whether we do not get farther with the problems of metaphysics by assuming that the objects must conform to our cognition”

The views of the fathers of quantum mechanics

“[T]he finite magnitude of the quantum of action prevents an altogether sharp distinction being made between the phenomenon and the agency by which it is being observed.”

“We meet here in a new light the old truth that in our description of nature the purpose is not to disclose the real essence of the phenomena but only to track down, so far as it is possible, relations between the manifold aspects of our experience.”

The views of the fathers of quantum mechanics



We "objectivate" a statement if we claim that its content does not depend on the conditions under which it can be verified. Practical realism assumes that there are statements that can be objectivated and that in fact the largest part of our experience in daily life consists of such statements. Dogmatic realism claims that there are no statements concerning the material world that cannot be objectivated. Practical realism has always been and will always be an essential part of natural science. Dogmatic realism, however, is, as we see it now, not a necessary condition for natural science. [...] Metaphysical realism goes one step further than dogmatic realism by saying that "the things really exist."

The views of the fathers of quantum mechanics

“[T]here remains still in the new kind of theory an objective reality, inasmuch as these theories deny any possibility for the observer to influence the results of a measurement, once the experimental arrangement is chosen. Therefore particular qualities of an individual observer do not enter the conceptual framework of the theory.”

The Copenhagen interpretation in reverse

Copenhagen interpretation

Quantum mechanics
forced upon us by experiment

Epistemic interpretation

The present approach

Epistemic assumptions
”forced upon us” by the
forms of cognition

Epistemic formalism

Quantum mechanics

Dirac equation
Entropy
Gauge principle
Pauli’s exclusion principle
...

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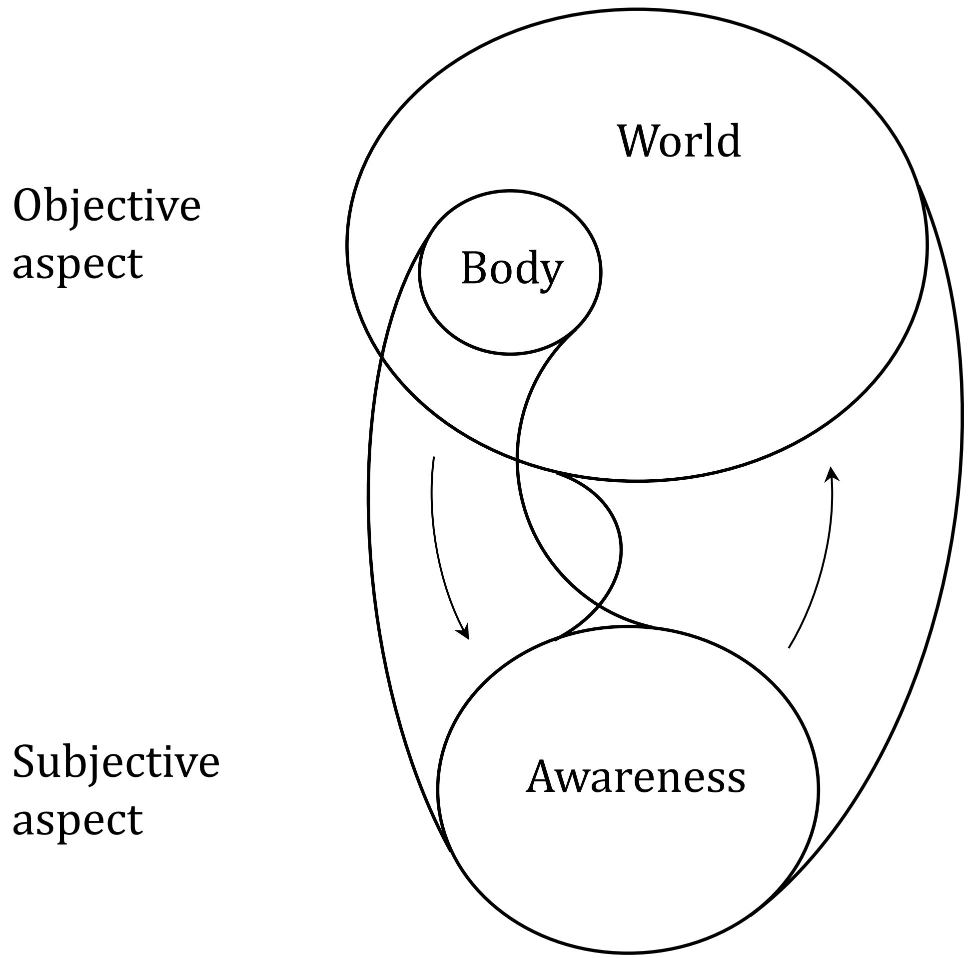
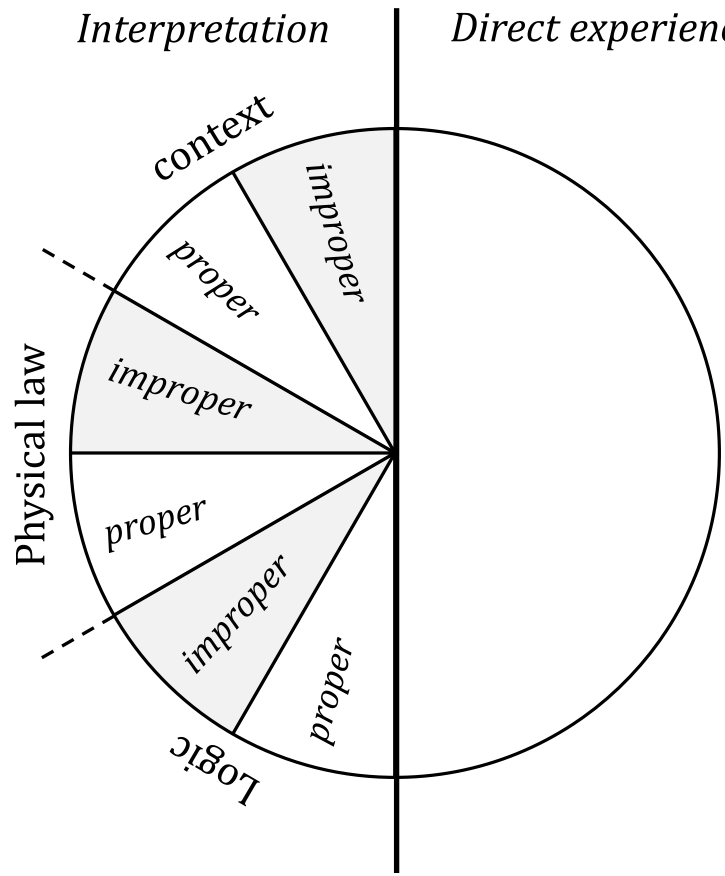
Intertwined dualism



Assumption of detailed materialism Each detail of a state of awareness corresponds to a detail of the state of objects in the body.

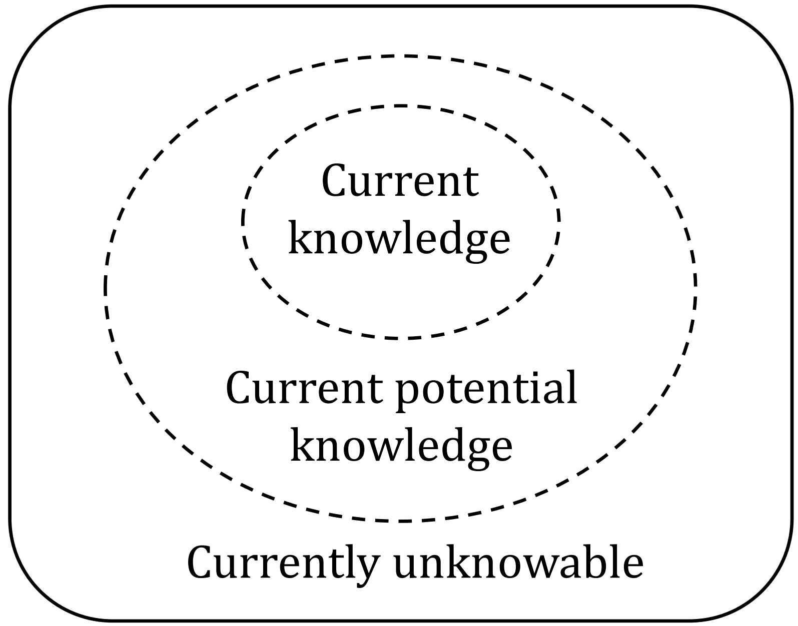
Knowledge

Knowledge: Aware perceptions with proper interpretation



We have to assume a transcendent distinction between proper and improper interpretations: ”The truth is out there”

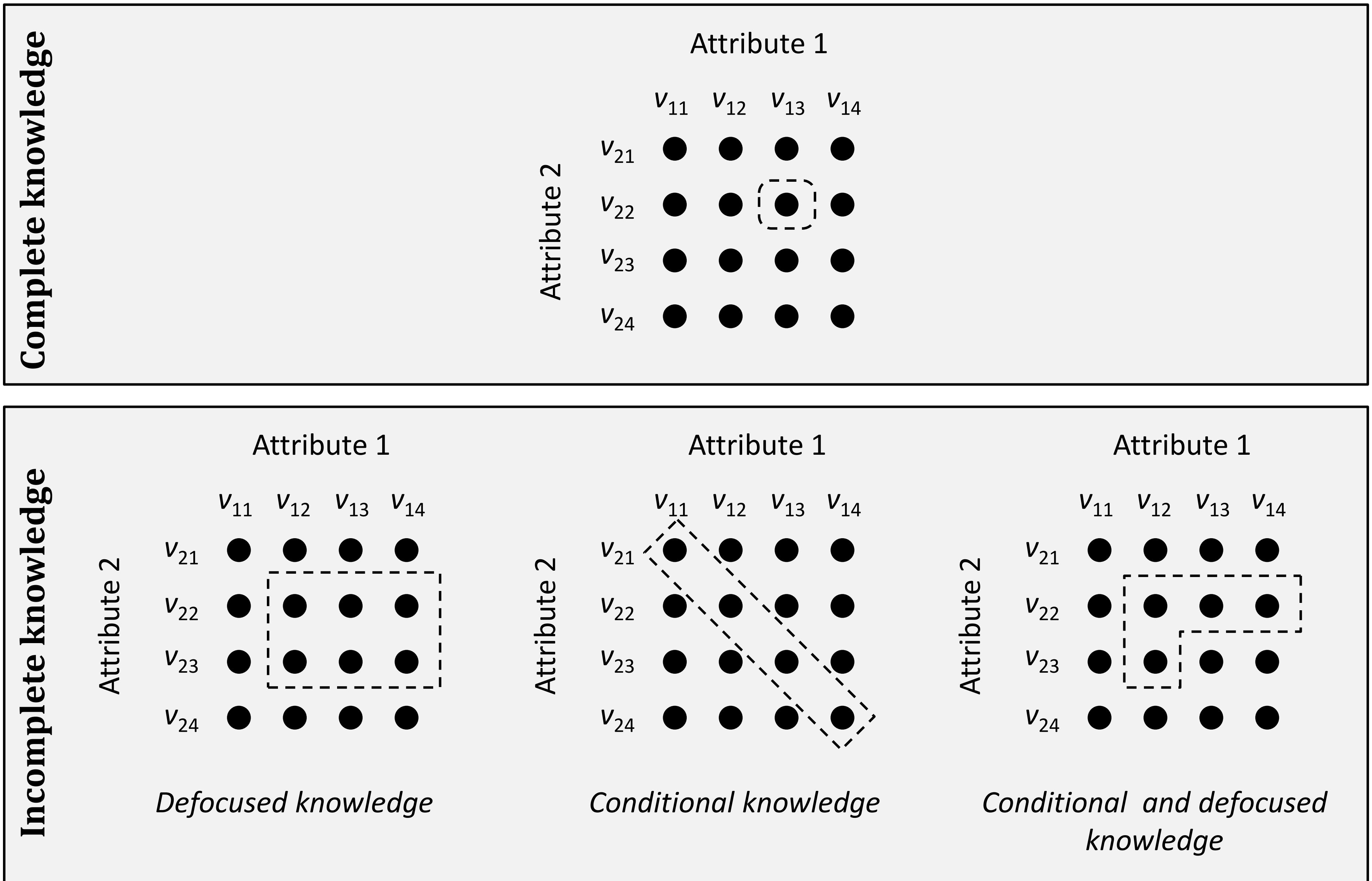
Knowledge and potential knowledge



Unknowable (not part of the world)

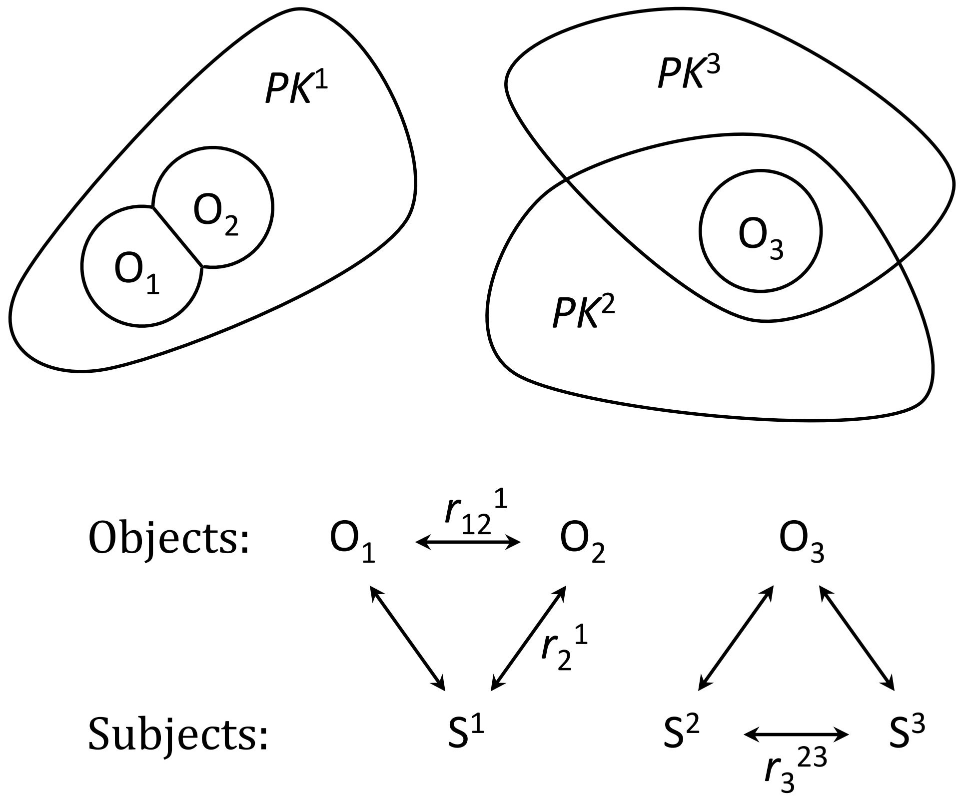
Knowledge	Those present aware perceptions that we interpret properly at (virtually) the same moment
Potential knowledge	Knowledge based on present perceptions that we potentially may become aware of and interpret properly now or later

Complete and incomplete knowledge



Individual and collective knowledge

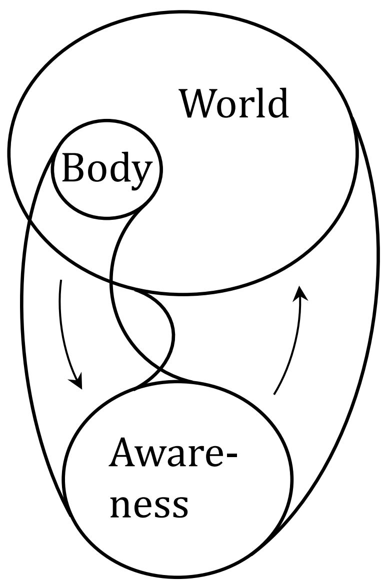
Individual subjects k has potential knowledge PK^k about objects O_l

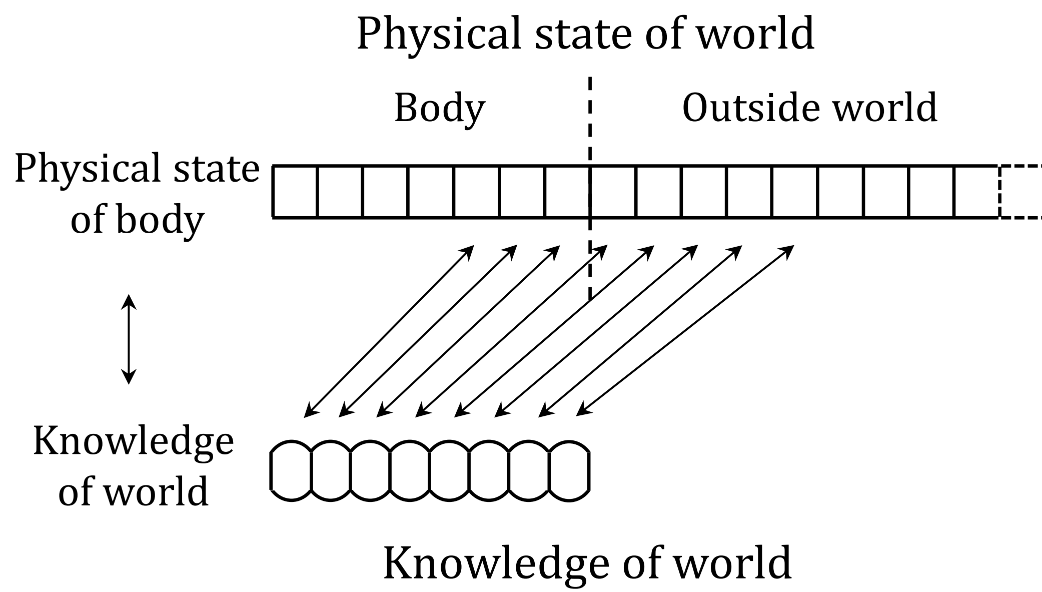


Symmetric relations between objects in the objective aspect and individuals in the subjective aspect

Knowledge is incomplete

Statement There is something currently unknowable



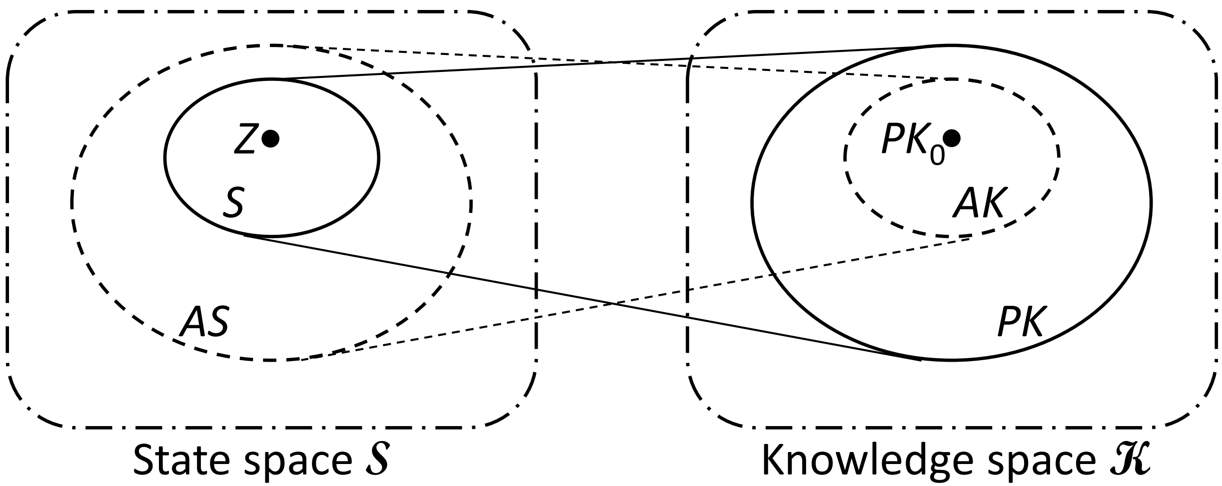


Motivation
The bodies of aware observers are a proper subset of all objects in the world, since we must leave room for the internal process of interpretation to speak about known objects at all. According to detailed materialism, a complete knowledge about the world would then be a proper subset of itself. This is possible only if the world is fractal. The seeming existence of elementary particles contradicts this possibility. Thus knowledge must be incomplete at any given time.

The physical state

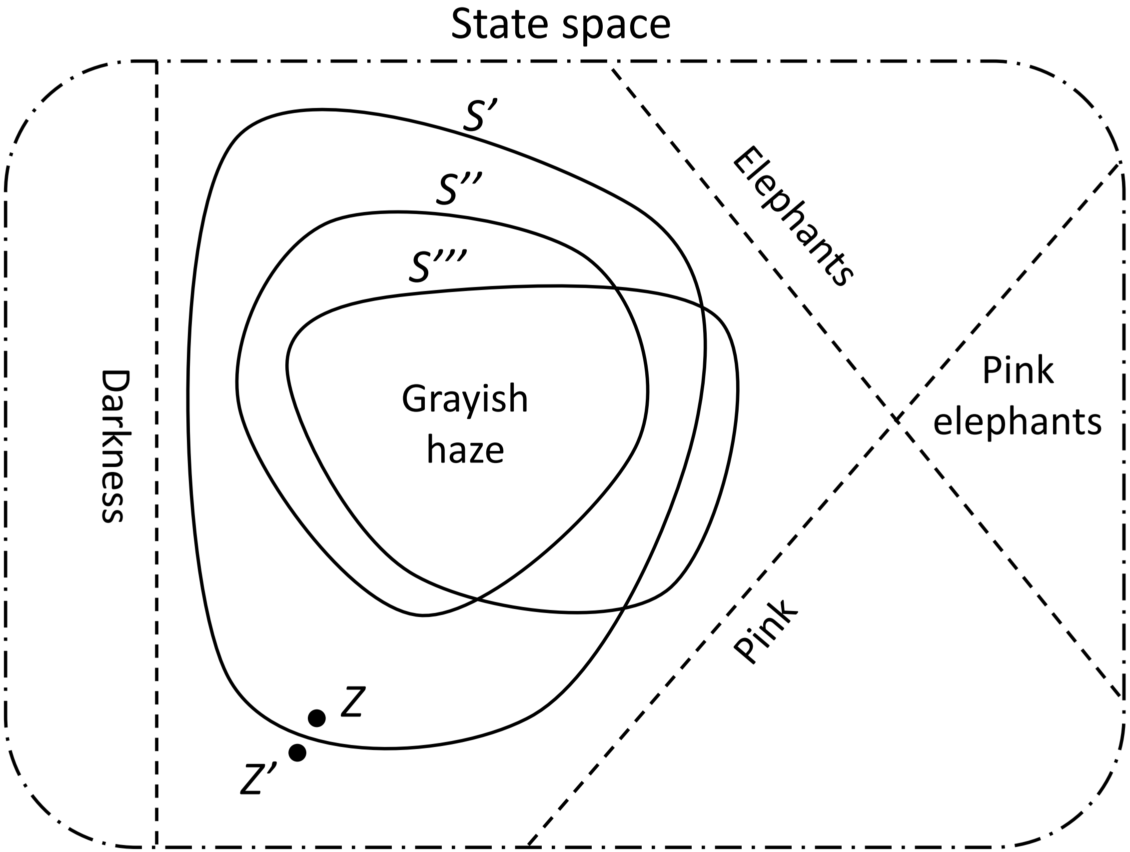
Each state of complete knowledge about the world corresponds to an element *Z* in state space *S*. Such an element corresponds to precise knowledge about all internal and relational attributes of all elementary particles.

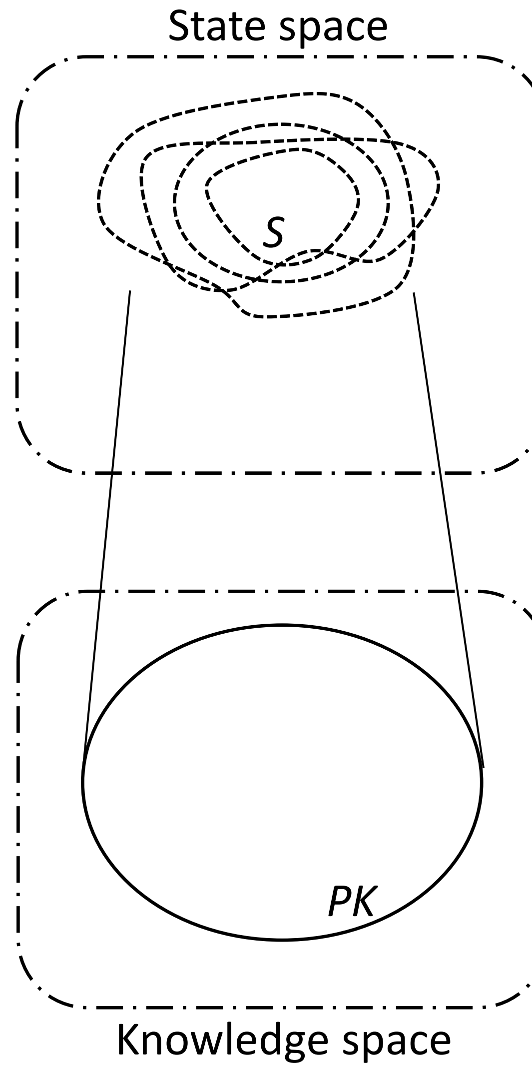
The physical state *S* is the union of all *Z* that cannot be excluded by the current collective potential knowledge *PK*.



PK_0 = Rudimentary awareness 'There is something'
AK = Current aware knowledge

Knowability of the physical state

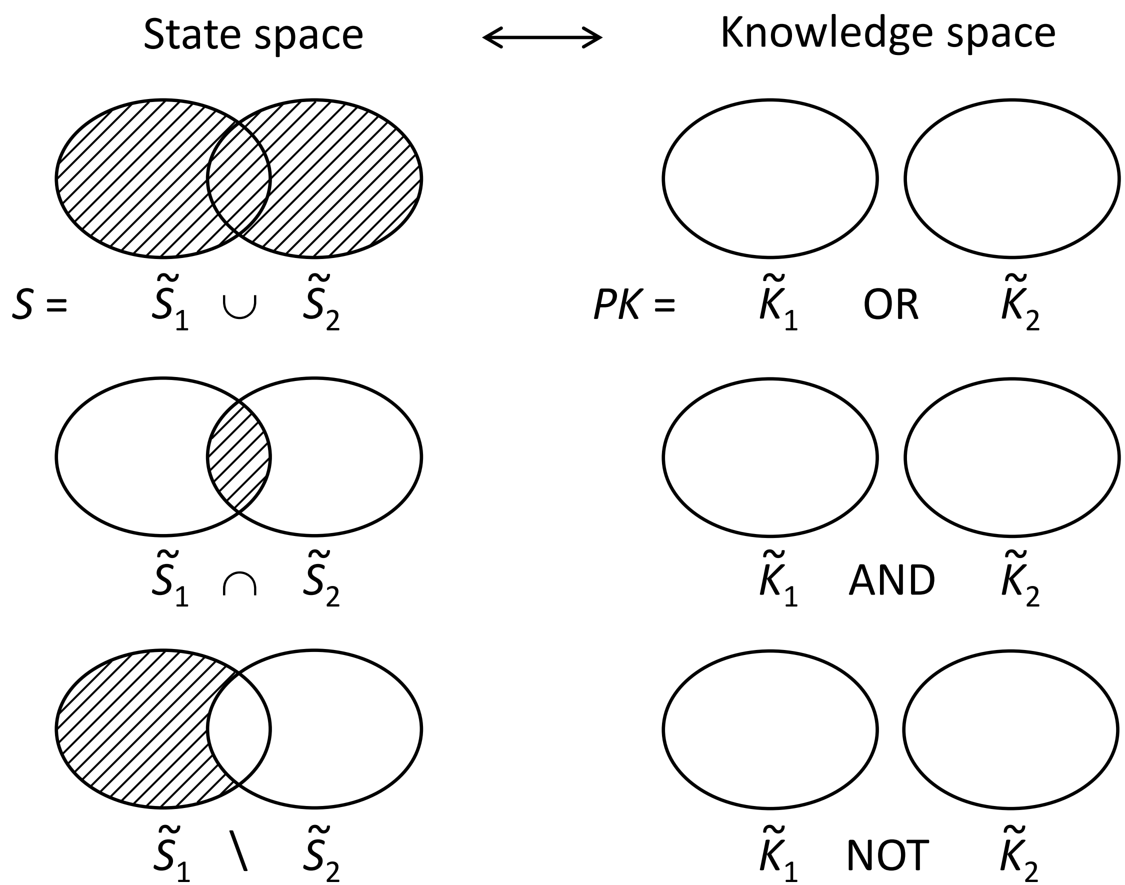




To pinpoint the boundary of *S* exactly we have to be able to distinguish between two exact states *Z* and *Z’* at either side of this boundary. The fundamental incompleteness of knowledge makes this impossible.

State space and knowledge space

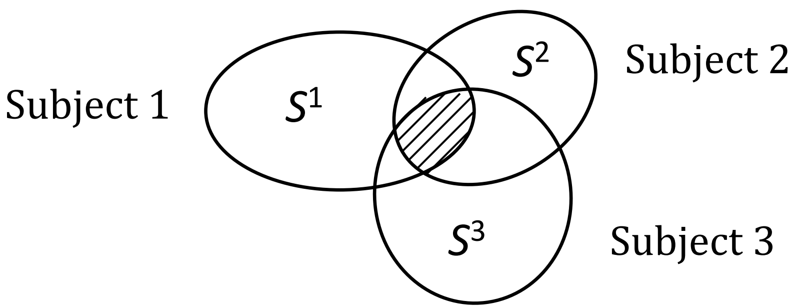
The physical state given two logically related items of knowledge

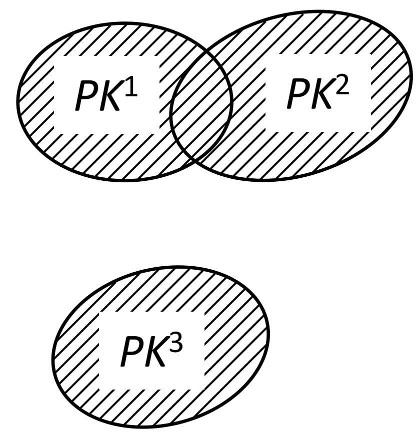


State space and knowledge space

Individual and collective states

State space $\longleftrightarrow$ Knowledge space



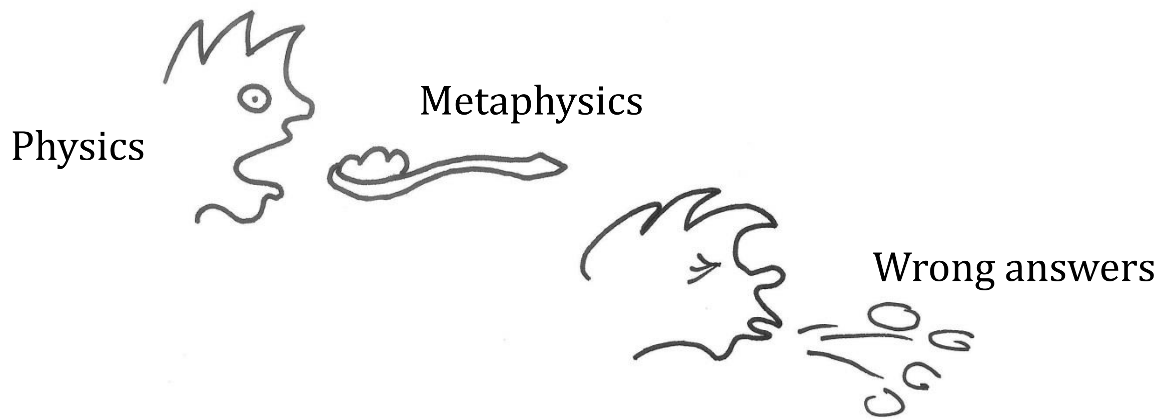


$$S = S^1 \cap S^2 \cap S^3 \qquad PK = PK^1 \cup PK^2 \cup PK^3$$

Each individual state S^k must overlap
all the others – otherwise the knowledge
of two subjects contradict each other!

Guiding principles for physical law I

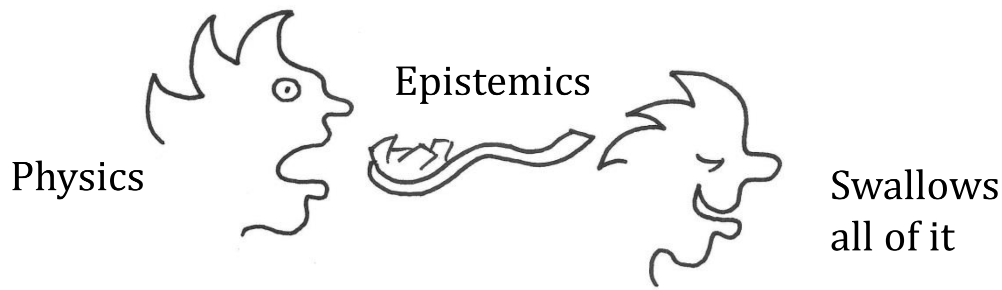
Epistemic minimalism
Physics spits out wrong answers if we try to feed it with metaphysics



Metaphysics = entities or distinctions that are impossible to observe in principle
(Objective Newtonian orbits, angular momentum of spherical objects,
interchange of identical particles, …)

Guiding principles for physical law I

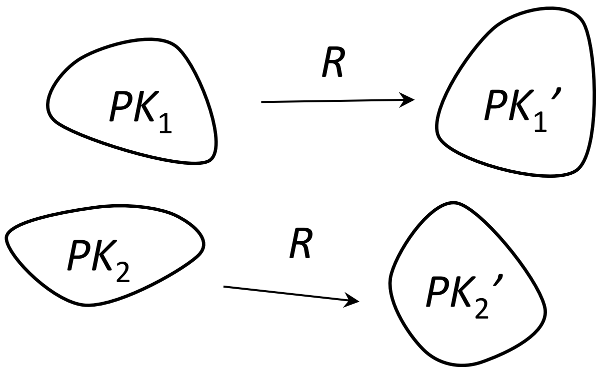
Epistemic completeness
Physics need and swallow all kinds of perceptions and distinctions between perceptions

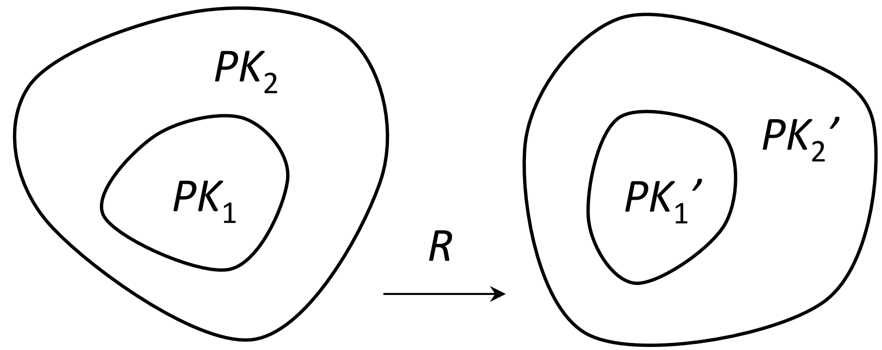


Epistemic minimalism + *Epistemic completeness* = *Epistemic closure*

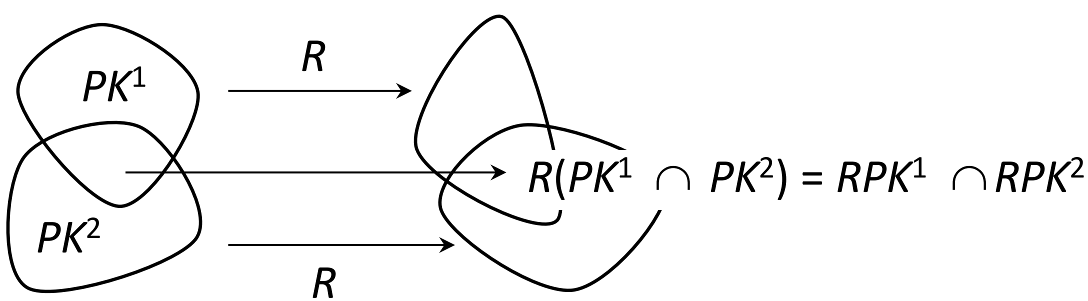
Guiding principles for physical law II

Collective epistemic invariance
The same evolution rule *R* applies regardless the content and the amount of potential knowledge *PK*





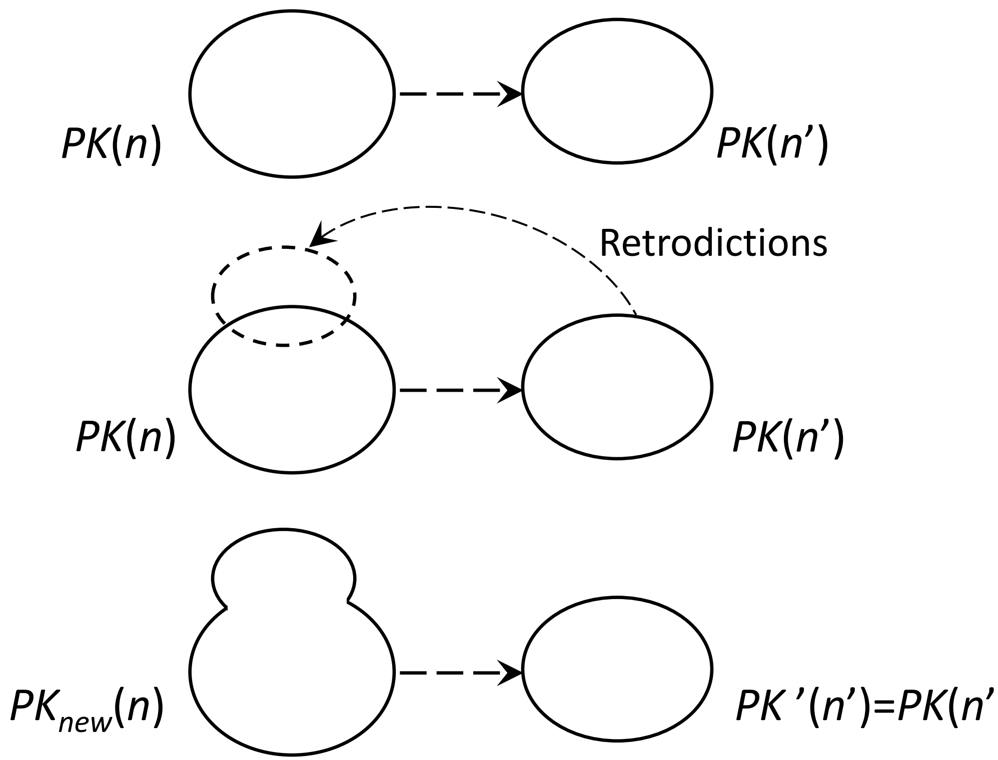
Individual epistemic invariance
The same evolution rule *R* applies to an object *k* regardless who possesses the individual potential knowledge PK^j_k about *k*. This means that R must be invariant to a change of perspective from one subject to another, leading the requirement of Lorentz invarance.



Guiding principles for physical law III

Epistemic consistency
A) Retrodictions about the past from present knowledge should never contradict memories of the past.

B) Retrodictions about the past that are made possible by new knowledge acquired at present should never increase the knowledge about the past in such a way that this expanded state of knowledge would have evolved to a different present state of knowledge than that we actually have

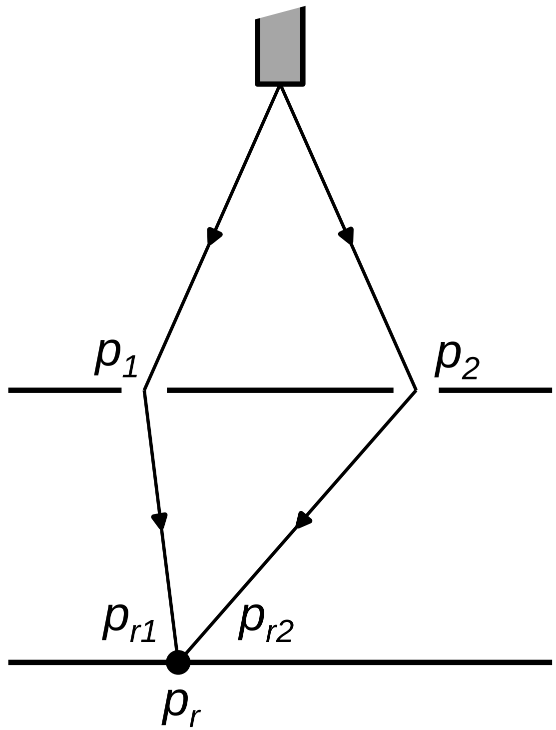


Some consequences of the guiding principles I

Consequences of epistemic closure

- The evolution *R* cannot act on each exact states *Z* individually, but must act on states that may correspond to a state of potential knowledge, such as the physical state S.
- Perceivable matter is fermionic – obeys Pauli's exclusion principle.
- To treat the exchange of identical particles as a different state gives rise to wrong statistics (Maxwell-Boltzmann rather tham Fermi-Dirac or Bose-Einstein)
- Spherically symmetric states must have zero angular momentum.
- Physical law cannot be parity invariant.
- There must be a finite highest possible speed (*c*) that all subjects agree about (Individual epistemic invariance is also needed)
- Physical law is rotationally and translationally invariant (both spatially and temporally)
- There must be interference-like phenomena (see next slide).

Some consequences of the guiding principles - towards interference



Path information belongs to potential knowledge

$p_r = p_1 p_{r1} + p_2 p_{r2}$

Path information does not belong to potential knowledge

$p_r \neq p_1 p_{r1} + p_2 p_{r2}$

Some consequences of the guiding principles II

Consequences of epistemic invariance

- Reductionism
- The equivalence principle (see next slide)

Consequences of individual epistemic invariance

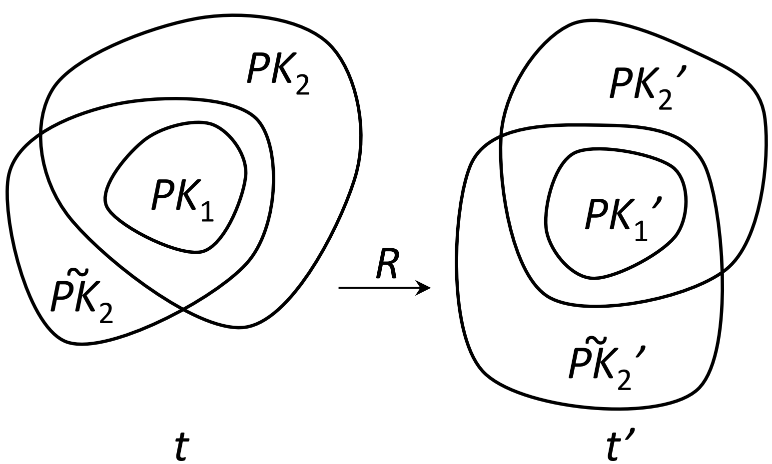
- Lorentz invariance
  (Epistemic closure in the form of epistemic minimalism is also needed)

Consequences of epistemic consistency

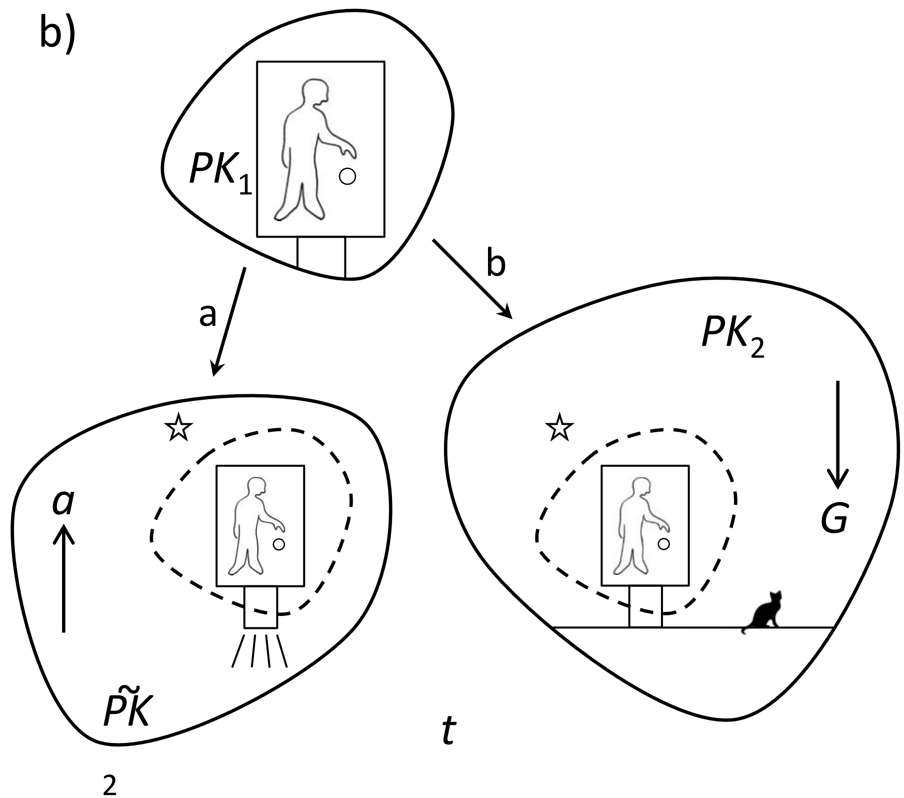
- Wave function collapse without observation (see second next slide)

Some consequences of the guiding principles
- the equivalence principle

a)



b)



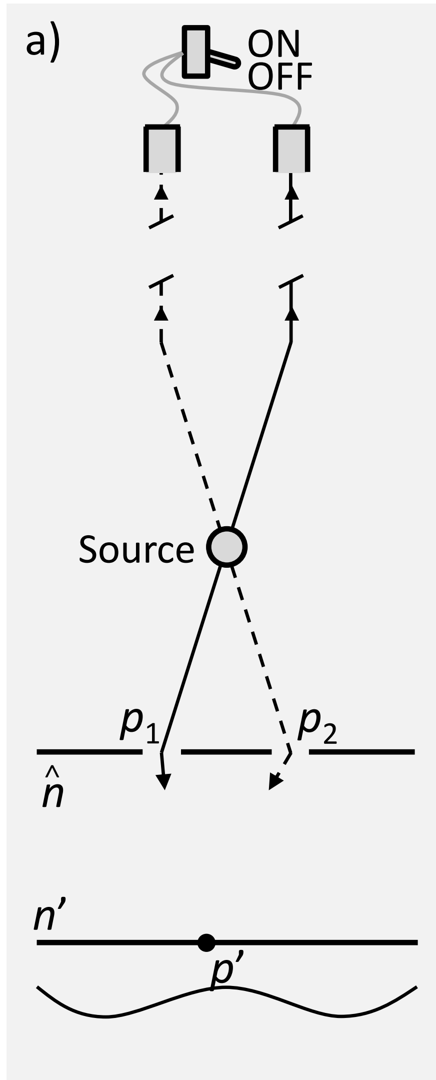
Some consequences of the guiding principles
- wave function collapse without observation

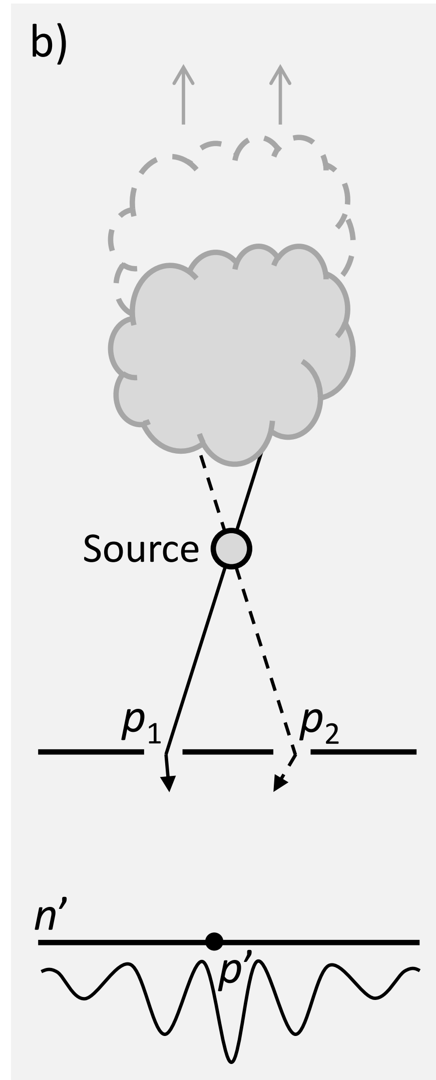
The particle travelling away from the slits is deflected into one of two detectors that may be far away and may or may not be turned on.

Two particles are emitted in opposite directions

By epistemic consistency, path information must be gained at the passing of the slits even if nobody observes the passing.

This corresponds to the collapse of the wave function and loss of interference pattern





The particle travelling away from the slits enters a ”cloud” that makes it impossible ever to tell its angle of entry into the cloud if it is detected later.

Two particles are emitted in opposite directions

This means that path Information about which slit the particle passes cannot be gained.

This corresponds to the survival of the wave function and appearance of interference pattern

Background

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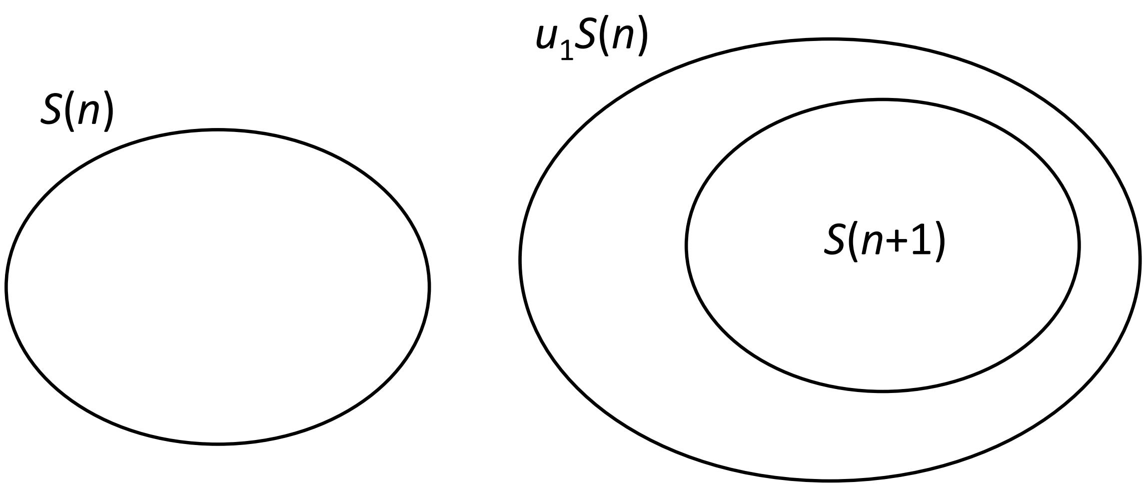
The evolution operator

Sequential time n

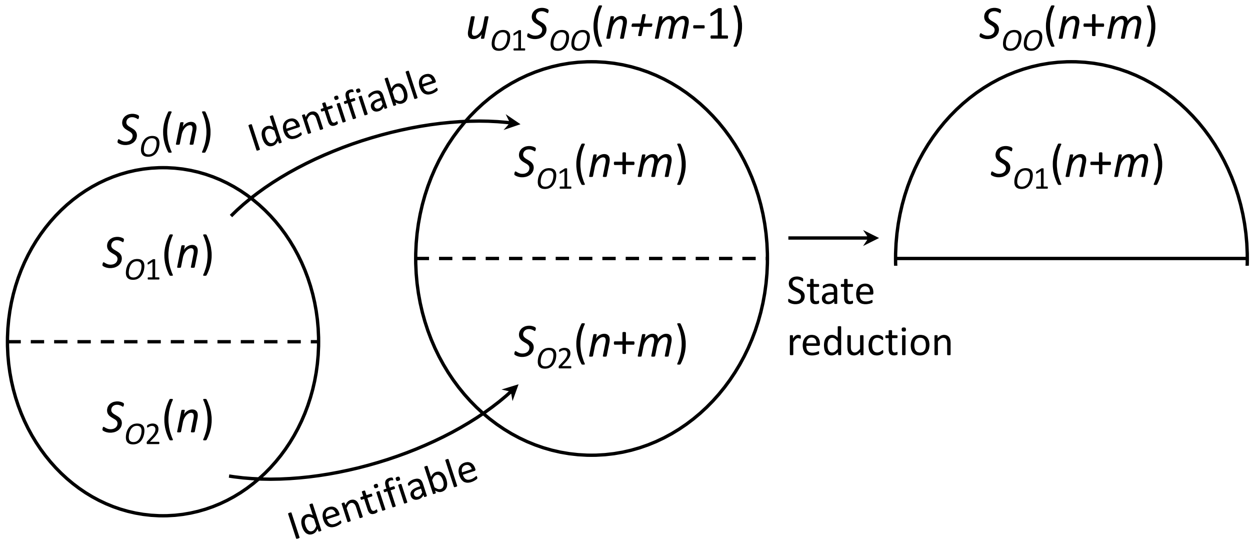
The physical state S is updated every time the potential knowledge PK^k of some subject k changes. We then write $n \to n + 1$ och $S(n) \to S(n+1)$.

The evolution operator u_1 is defined by the condition that $u_1 S(n)$ is the smallest possible set $C \subseteq S$ such that physical law dictates that $S(n + 1) \subseteq C$.



$S(n)$ and $S(n+1)$ subjectively distinct by definition $\Rightarrow$ $S(n) \cap u_1 S(n) = 0$

Alternatives S_{0j}



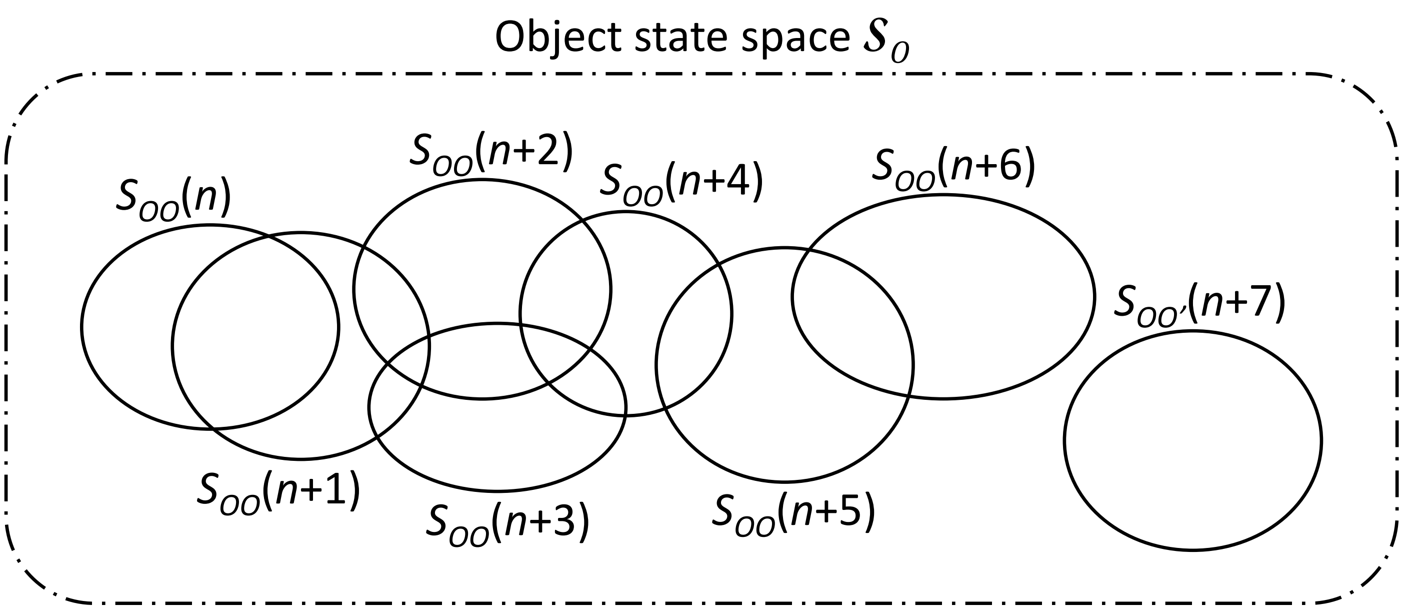
$S_{OO}(n) \subseteq \mathcal{S}_0$ is the union of all states Z_0 of complete knowledge of an object O that are not excluded by the knowledge about O.

The alternatives S_{0j} corresponds to states of the object that may arise if we learn more about it

”Linear” evolution of alternatives: $u_{O1}S_{OO} = u_{O1}\bigcup_j S_{Oj} = \bigcup_j u_{O1}S_{Oj}$

Identifiability

The object *O* for which some alternative may turn out to be true at some future time must be possible to follow through time in such a way that it can be considered ’the same’ at different times.

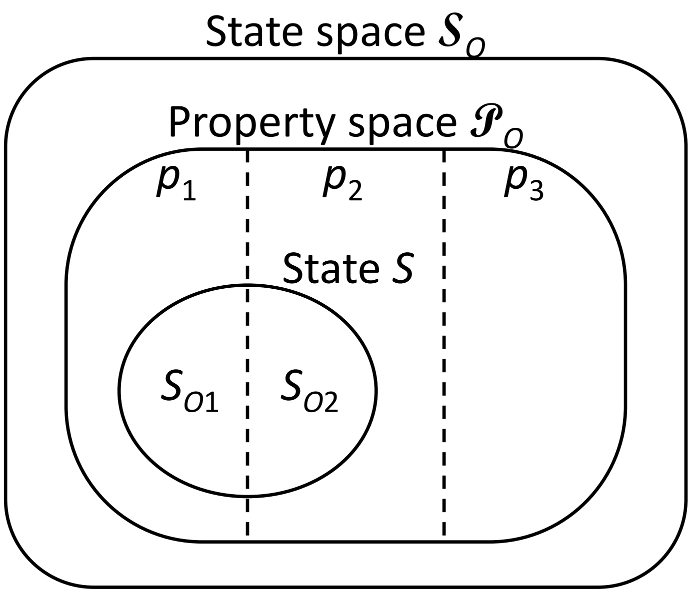


O is identifiable in the time interval [*n*, *n*+6]

Properties

The realization of an alternative can often be seen as the observation of a particular value of some property.

A property P is a statement about the attributes of a set of objects.
We may express its value p as $p = f(\{v_{il}\})$ where v_{il} is the value of attribute A_i och object O_l.



The property space P_O is the union of all states Z_O of exact knowledge for which there are objects such that property P can be defined for the object O.

Levels of knowability of a set of alternatives

1) **It will never become known which alternative is true.**
No property value p_j that corresponds to the alternative S_{0j} in the set $\{S_{0j}\}$ can ever be observed given the present physical object state $S_{OO}(n)$.

2) **It may become known which alternative is true.**
There is a time $\hat{n} > n$ such that it is possible that such a property value p_j is observed at some time $n' \geq \hat{n}$, so that $S_{OO}(n') = S_{Oj}(n')$, but it is not dictated by physical law that this will happen. We let $\hat{n}$ be the smallest possible such time.

3) **It will become known which alternative is true.**
There is also a time $\check{n} > n$ such that physical law dictates that one of the property values p_j will be observed at some time $n' \leq \check{n}$, so that $S_{OO}(n') = S_{Oj}(n')$. We let $\check{n}$ be the smallest possible such `deadline for decision'.

Complete sets of future alternatives

A set of alternatives $\{S_{0j}\}$ is a complete set of future alternatives if and only if

- Each property value p_j that corresponds to an alternative S_{0j} in $\{S_j\}$ **may** be observed at some time $\hat{n} \geq n+1$, given the present physical state $S_{00}(n)$.

- One of the property values in the set $\{p_j\}$ that corresponds to $\{S_{0j}\}$ **will** be observed at some time $\hat{n} \geq n+1$.

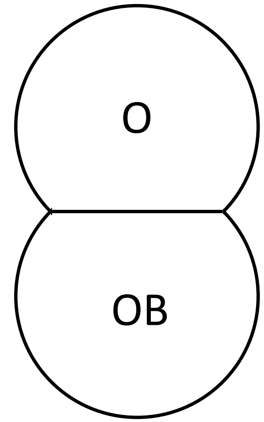
- The property P and the set of possible values $\{p_j\}$ are defined in such a way that if p_j is observed at time $\hat{n}$, then we can **never** observe any other property value in the set $\{p_j\}$ at a later time.

The future alternative S_{0j} is the union of those states S_{00} which has the value p_j of property P, or will have it as S_{00} evolves.

Observational setups

Necessary objects in any observation

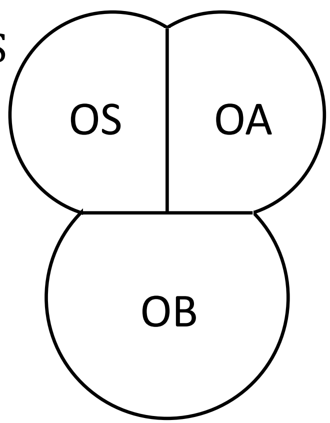
Observed
Object O



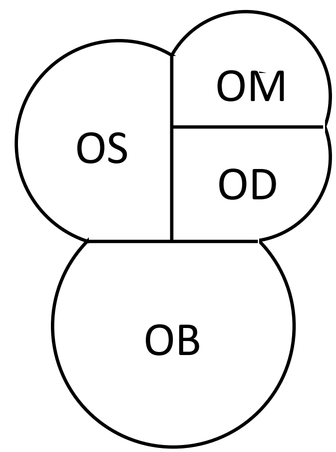
Body OB of
observer

Objects involved in an experiment

The specimen OS
for which a set
of alternatives is
defined



Apparatus OA
used to decide
the corresponding
property of
the specimen



A change of the
detector OD defines
the outcome of the
experiment when
the specimen is a
quasiobject.

Physics formulates laws of nature in terms of the behaviour of specimens

The observational context C

- C is the potential knowledge contained in the state S_{00} of the observed object O, together with a sequence of complete sets of alternatives $\{S_j\}$, $\{S_j'\}$, …$\{S_j^{(F)}\}$ that corresponds to a series of properties P, P', …, $P^{(F)}$ observed in sequence.

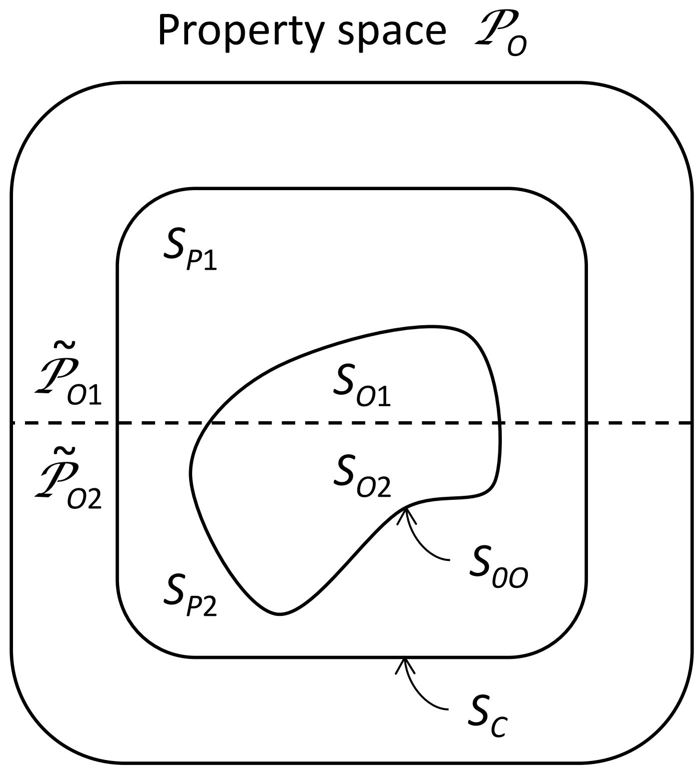
- These properties are defined for a specimen OS that is part of O.

- The knowability level associated with each property should be 1 or 3 at the start of the experiment , and the knowability level associated with $P^{(F)}$ should be 3.

A context is initiated at some given time n, which is a point of no return: after that the sequence of properties **will** be observed according to the above.

The observational setup in state space

$\tilde{\mathcal{P}}_{Oj}$ is the union of all states for which the value of P is or will be p_j.



S_{Oj} is a future alternative defined within the observational context C.

The property value state $S_{Pj} \leftrightarrow$ The knowledge about the nature of the specimen OS, together with knowledge about the value p_j of one of its unknown properties P.

The contextual state S_C

Consider an observational context C which is initiated at time n in which a series of properties P, P', …, $P^{(F)}$ of the specimen OS is observed at the times $n+m$, $n+m'$, …, $n+m^{(F)}$.

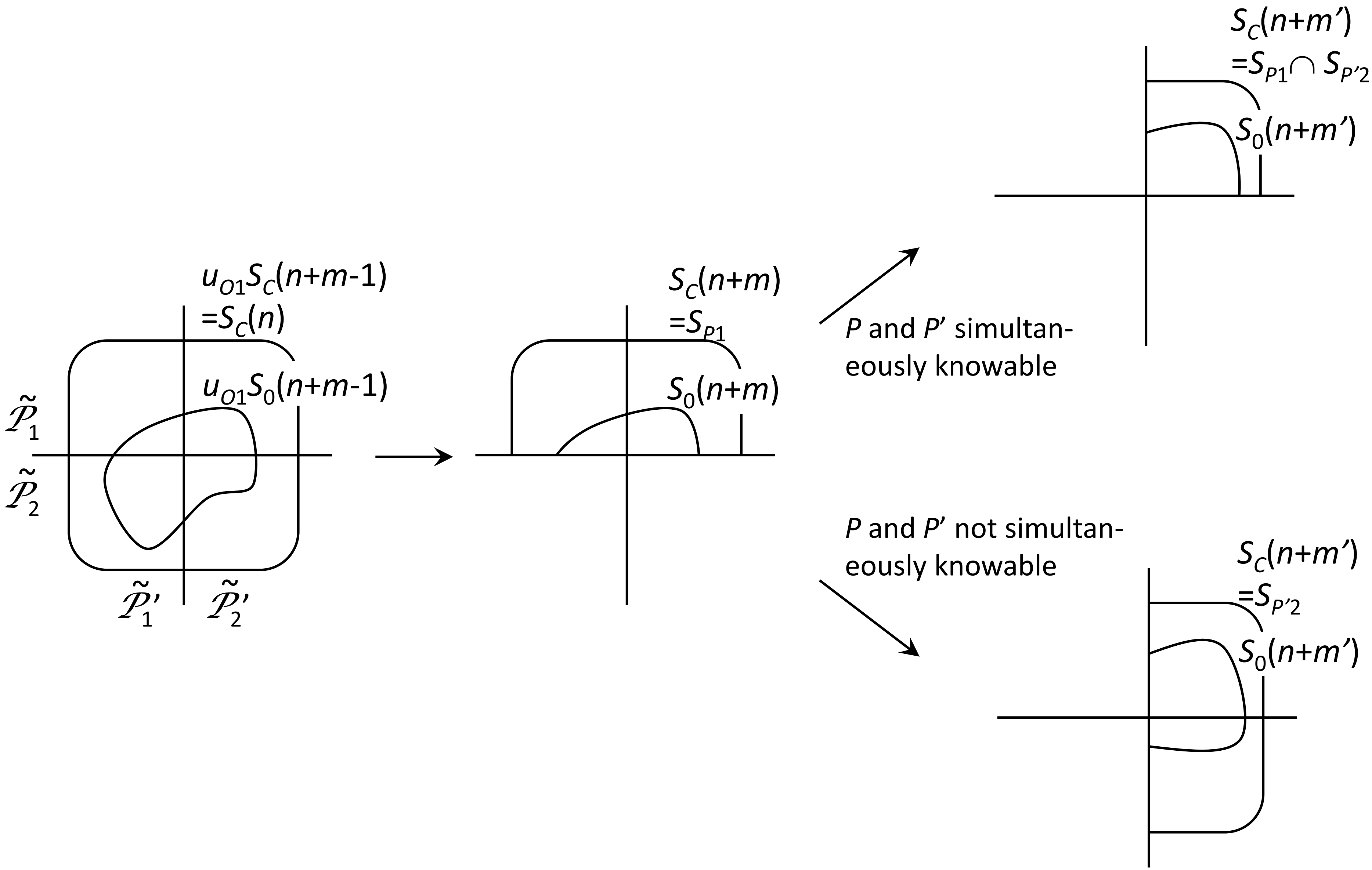
Then $S_C(n')$ is defined for $n \leq n' \leq n+m^{(F)}$ and corresponds to the potential knowledge of these properties at time n', in addition to prior knowledge about the nature of the specimen.

We always have $S_C(n) = \bigcup_j S_{Pj} = \bigcup_j S_{P'j} = \bigcup_j S_{P^{(F)}j}$

S_C is a contextual state since it corresponds to the knowledge of OS as seen via the apparatus OA, arranged to record a set of properties, predefined by the observer with body OB. That is, OA and OB are necessary to define S_C.

The observation sequence

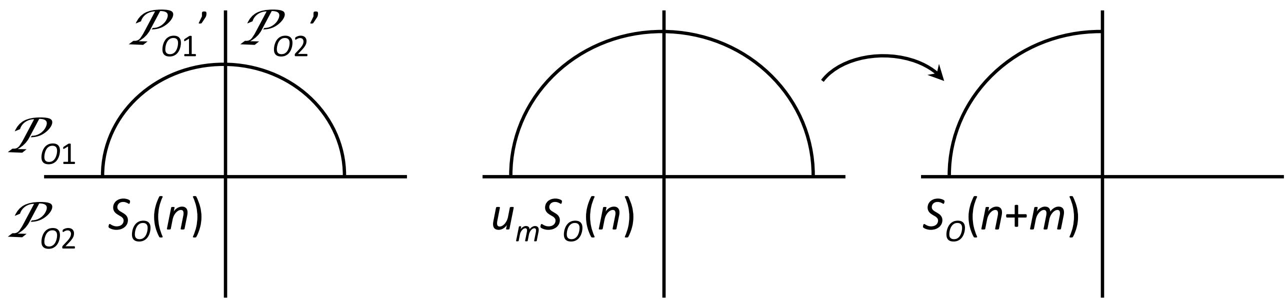
When two properties P and P' with binary values are observed



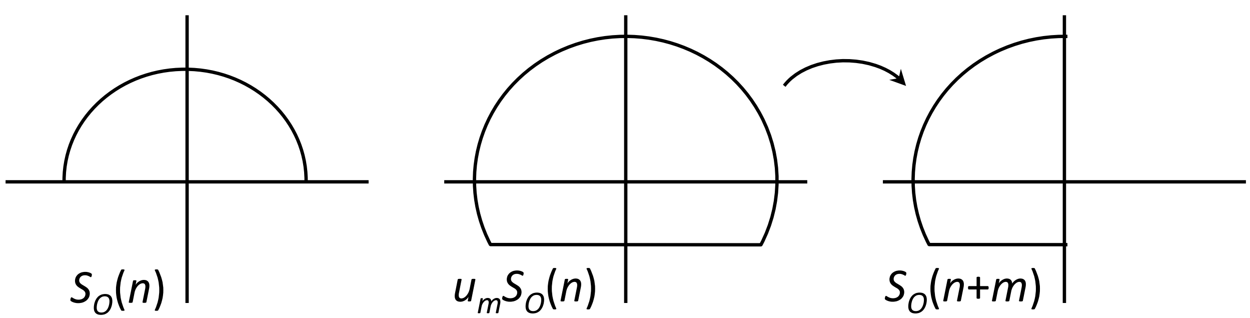
Simultaneous knowability

Knowledge is incomplete ⇒ There are pairs of properties *P* and *P'* whose values are not simultaneously knowable

P and *P'* simultaneously knowable



P and *P'* not simultaneously knowable



Background

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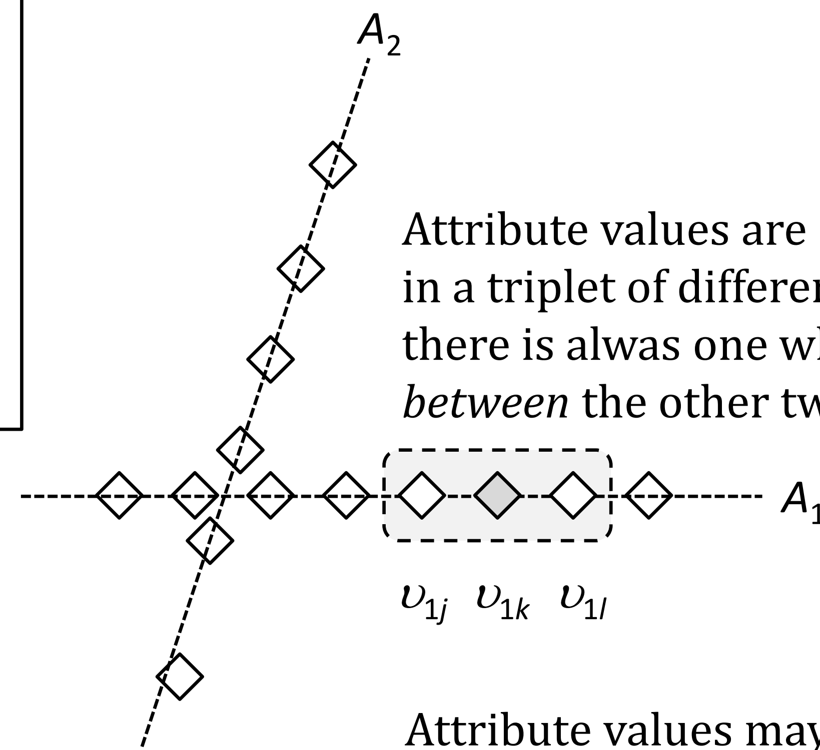
Structure of state space

Assumption All knowledge can be Expressed in terms of objects, attributes and their values

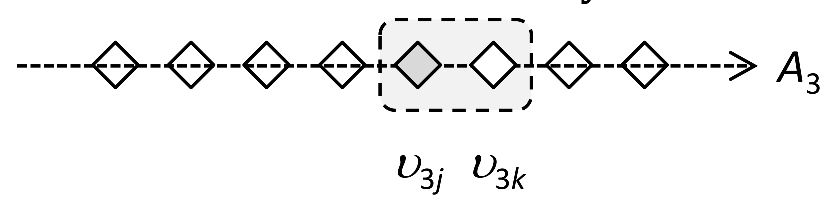
The axes defined by the attributes A_j span state space S. Each object that may be present is allowed its own set of such axes.

Attributes may be internal or relational.

Attribute values are ordered: in a triplet of different values, there is alwas one which is in *between* the other two.



Attribute values may be directed: then, in a pair of different values, one always *succeed* another.



Measure V on state space

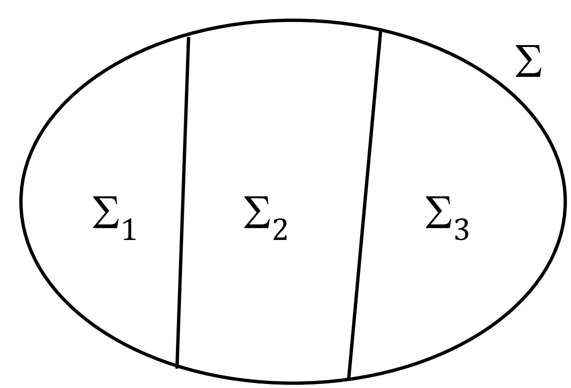
The attribute value space $\boldsymbol{S}(A, \upsilon) \subseteq \boldsymbol{S}$ is the set of exact states Z for which there is at least one object for which the attribute A is defined and for which its value is υ.

The state space volume V:

$V[\boldsymbol{S}(A, \upsilon)] = V[\boldsymbol{S}(A, \upsilon')]$ for any A and any pair of values (υ, υ') allowed by physial law
$V[\Sigma_1 \cup \Sigma_2] = V[\Sigma_1] + V[\Sigma_2]$ for any pair of disjoint sets (Σ_1, Σ_2) in state space $\boldsymbol{S}$.
$V[Z] = 1$

The relative volume v:

$v_j \equiv V[\Sigma_j]/V[\Sigma]$ for any partition $\Sigma = \bigcup_j \Sigma_j$



Existence of probability

An epistemically meaningful notion of probability can only be assigned

- to subjectivley preconceived alternatives S_j applying to a system O with initial state S_0.
- to realizable such alternatives.
- to future alternatives.
- to a complete set of future alternatives at knowability level 3.
- if it is knowable *a priori*, before the trial is carried out. This means that the set of future alternatives must be repeatable, or they must have a preconceived symmetry.

Probabilities cannot be associated with all non-deterministic state reductions $S_0(n+1) \subset u_1 S_0(n)$. They cannot be associated with reductions of the universal state $S(n)$, since an external observer that predefines alternatives is needed.

When it exists, probability is relative volume v

Let q_j be the probability for the future alternative S_{Oj}. In general we may write

$q_j = f[S_{Oj}, \{S_{Oj'}\}, S_{OO}, S]$.

However, the object O to which q_j apply must be isolated from the environment if q_j is obtained by repeated preceding trials. If q_j is deduced by symmetry, everyting that affects these symmetries must be included in O. In either case we may drop S as an argument. We may also drop $\{S_{j'}\}$ if we exclude 'mental influences' on q_j.

The axioms of probability can then be written

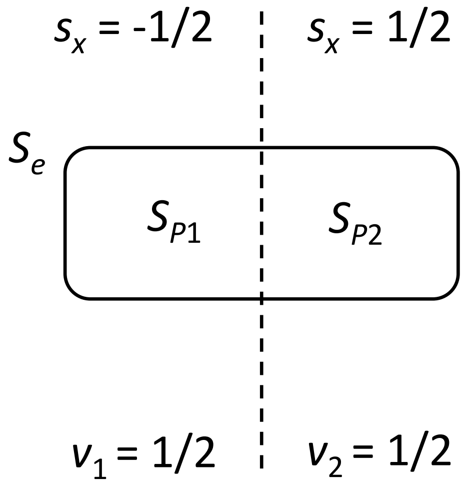
$f[S_{Oj}, S_{OO}] \geq 0$
$f[S_{OO}, S_{OO}] = 1$
$f[S_{Oj} \cup S_{Oj'}, S_{OO}] = f[S_{Oj}, S_{OO}] + f[S_{Oj'}, S_{OO}]$

These relations are always fulfilled if and only if

$q_j = v_j = V[S_{Oj}]/V[S_{OO}]$

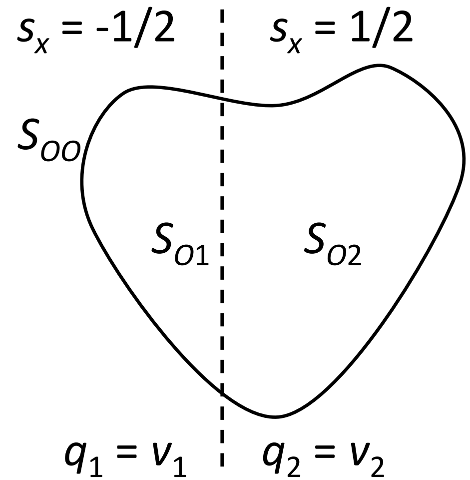
Probability is a macroscopic quantity

Spin state of electron
(with x-axis defined)



S_e is the object state of the electron. There are equally many states of the world consistent with either spin value: $V[S_{P1}] = V[S_{P2}]$. $v_1 = v_2 = ½$ is not the probability to observe these values

State of observational setup
to measure spin of the electron



S_{OO} is the object state of the observational setup, including electron OS and apparatus OA. The knowledge about O of the observer with body OB may correspond to different probabilities $q_1 = v_1$ and $q_2 = v_2$ to observe either spin value.

Background

The basic picture

The measurement process

Probability

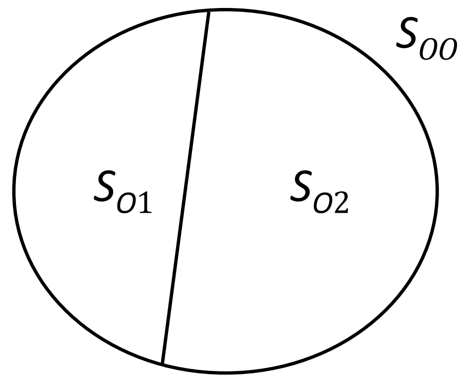
Elements of quantum mechanics

State representations

The state S	A set in state space $\boldsymbol{S}$ where the elements are hypothetical states Z of complete knowledge about the world
A state representation $\bar{S}$	A symbolic or algebraic encoding of the knowledge contained in the state S.
$\bar{S} \hookrightarrow S$	$\bar{S}$ is a complete representation of S; it represents all knowledge contained in S.
$\bar{S} \rightharpoondown S$	$\bar{S}$ is a partial representation of S; it represents some knowledge of interest contained in S.

Representations of object states S_{OO} with alternatives

$\{S_{Oj}\}$ is a complete set
of future alternatives



$v_1 < v_2$

$$\overline{S}_O \equiv \begin{bmatrix} S_{O1} & S_{O2} \\ v_1 & v_2 \end{bmatrix} \hookrightarrow \ S_{OO}$$

By the definition of a complete set
of future alternatives, we have $u_{O1}v_j = v_j$
before one of them is realized.

$$\overline{u}_{O1}\,\overline{S}_{OO} = \begin{bmatrix} u_{O1}S_{O1} & u_{O1}S_{O2} \\ v_1 & v_2 \end{bmatrix} \hookrightarrow u_{O1}\ S_{OO}$$

(The representation is over-determined in the sense that v_j is a function of S_{Oj}.)

Proto-algebraic representation of S_{OO}

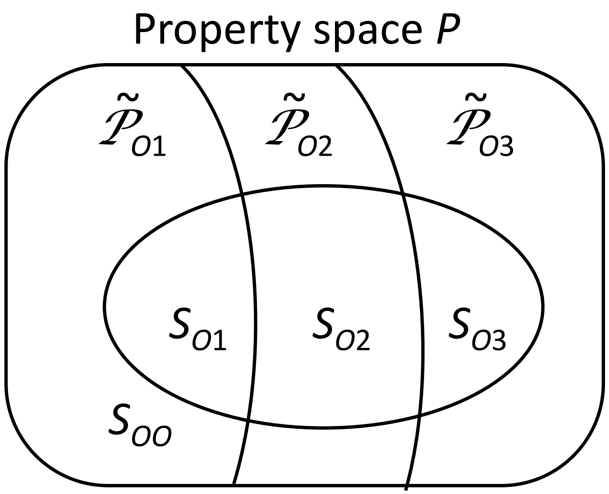
Formal notation without any predefined algebraic meaning attached to A+B and AB

$$\overline{S}_{OO} \equiv \begin{bmatrix} S_{O1} & S_{O2} \\ v_1 & v_2 \end{bmatrix} \quad \leftrightarrow \quad \overline{S}_{OO} = v_1 \overline{S}_{O1} + v_2 \overline{S}_{O2}$$

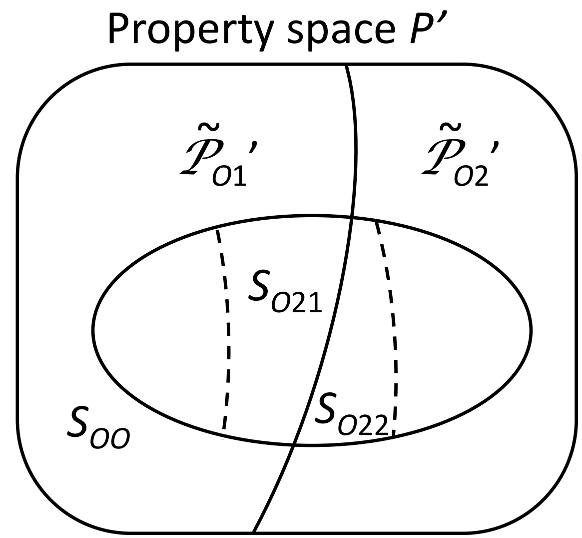
$$\overline{u}_{O1}\,\overline{S}_{OO} = \begin{bmatrix} u_{O1} S_{O1} & u_{O1} S_{O2} \\ v_1 & v_2 \end{bmatrix} \quad \leftrightarrow \quad u_{O1} \overline{S}_{OO} = v_1 u_{O1} \overline{S}_{O1} + v_2 u_{O1} \overline{S}_{O2}$$

The evolution operator u_{O1} becomes formally linear

Proto-algebraic representation of successive observations of properties *P* and *P*'.



Assume that *P* and *P*' are simultaneously knowable



(*P*, *P*') is a combined property with four possible values (p_1, p_1'), (p_2, p_1'), (p_2, p_2'), (p_3, p_2'), and corresponding complete set of future alternatives $S_{O1}, S_{O21}, S_{O22}, S_{O3}$.

$$\overline{S}_O \equiv \begin{bmatrix} S_{O1} & S_{O2} & S_{O3} \\ v_1 & v_2 & v_3 \end{bmatrix} \qquad \overline{S}_O \equiv \begin{bmatrix} S_{O1} & S_{O21} & S_{O22} & S_{O3} \\ v_1 & v_2 v_{21} & v_2 v_{22} & v_3 \end{bmatrix} \qquad \overline{S}_2 \equiv \begin{bmatrix} S_{O21} & S_{O22} \\ v_{21} & v_{22} \end{bmatrix}$$

with $v_j = V[S_{0j}]/V[S_{OO}]$ and $v_{2j} = V[S_{O2j}]/V[S_{O2}]$

Proto-algebraic representation of successive observations of properties P and P'.

Bars on top of alternatives to open up for further division of them in a subsequent observation

$$\overline{S}_{OO} = v_1\overline{S}_{O1} + v_2\overline{S}_{O2} + v_3\overline{S}_{O3} = v_1\overline{S}_{O1} + v_2\left(v_{21}\overline{S}_{O21} + v_{22}\overline{S}_{O22}\right) + v_3\overline{S}_{O3}$$
$$= v_1\overline{S}_{O1} + v_2 v_{21}\overline{S}_{O21} + v_2 v_{22}\overline{S}_{O22} + v_3\overline{S}_{O3}$$

The distributive law holds

Algebraic representation of the contextual state S_C

$$\overline{S}_C \equiv \begin{bmatrix} S_{P1} & S_{P2} \\ a_1 & a_2 \end{bmatrix} \qquad \leftrightarrow \qquad \overline{S}_C = a_1\overline{S}_{P1} + a_2\overline{S}_{P2}$$

Desiderata

1) The numbers a_i can be used to calculate the probability q_i to have $S_C = S_{Pi}$ after the observation of P, whenever q_i exists:

$v_i = f(a_i)$, (and $q_i = v_i$ whenever q_i exists)

2) We may define a linear evolution operator u_C that takes S_C from the instant just after one observation to the instant before next: $u_C S_C(n) = u_1 S_C(n+m\text{-}1)$:

$$\overline{u}_C\overline{S}_C = \overline{u}_C\left(a_1\overline{S}_{P1} + a_2\overline{S}_{P2}\right) = a_1\overline{u}_C\overline{S}_{P1} + a_2\overline{u}_C\overline{S}_{P2}$$

3) The distributive law holds for a_j and S_{Pj} :

$$a_1(a_2 + a_3)\ \overline{S}_{Pj} = (a_1a_2 + a_1a_3)\overline{S}_{Pj}$$
$$(a_1 + a_2)\ \overline{S}_{Pj} = a_1\overline{S}_{Pj} + a_2\overline{S}_{Pj}$$

4) The form of the representation is generally valid: it applies in all kinds of contexts, regardless the number of properties observed in succession and their knowability level, always respecting the principle of epistemic closure.

Algebraic representation of the contextual state S_C

Notes

The numbers a_i are not related to the relative volumes of the property value states S_{pi} which describe the specimen OS, but via desiderata 1) to the relative volumes of the future alternatives S_i of the entire (macroscopic) observational setup O, including the apparatus OA.

The representation S_C of S_C is even more ”contextual” than S_C itself, since it points explicitly viā a_j outwards from OS to the means OA by which we observe it.

$\bar{S}_C$ is a complete (over-determined) representation of S_C, but just a partial representation of O.

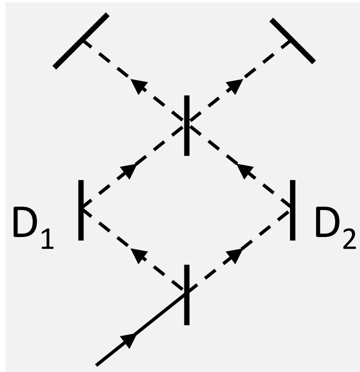
$$\bar{S}_C \hookrightarrow S_C$$
$$\bar{S}_C \rightharpoondown S_O$$

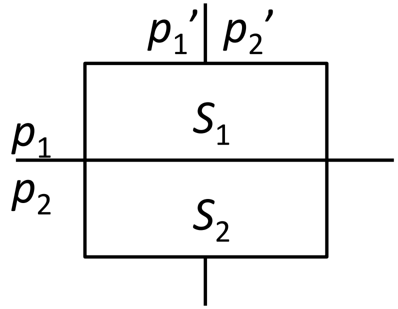
This means that the states S_O and $S_{O'}$ of different contexts may have the same representation: $\bar{S}_C \rightharpoondown S_O$ and $\bar{S}_C \rightharpoondown S_{O'}$.

Example: Mach-Zehnder contexts

Property P:
Which detector D_1 or D_2 will the photon hit?

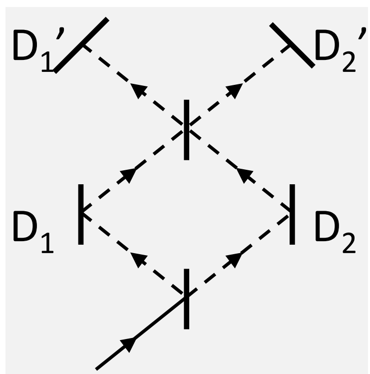
a)

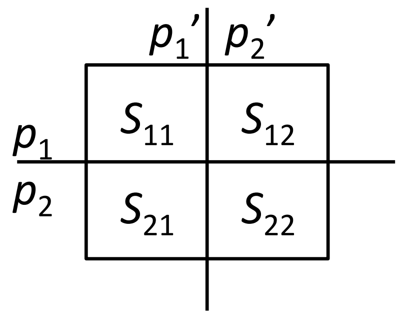




One property P observed within context.

b)

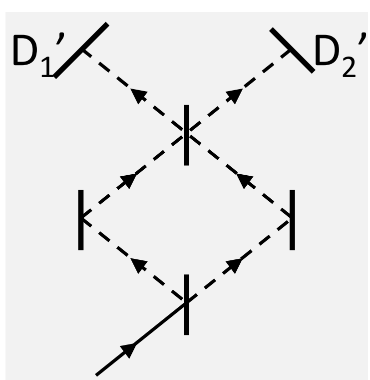


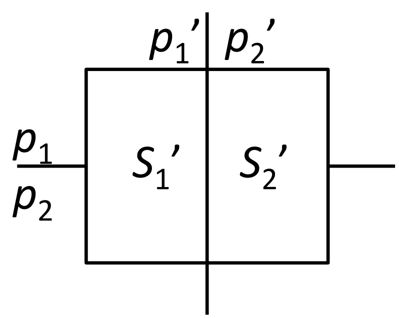


Two properties P and P' observed within context.

Property P':
Which detector D_1' or D_2' will the photon hit?

c)





Two properties P and P' 'observed' in context, but the value p_j of P outside potential knowledge (knowability level 1)

Algebraic representation of Mach-Zehnder contexts

P and *P*’ at knowability level 3 (case b)

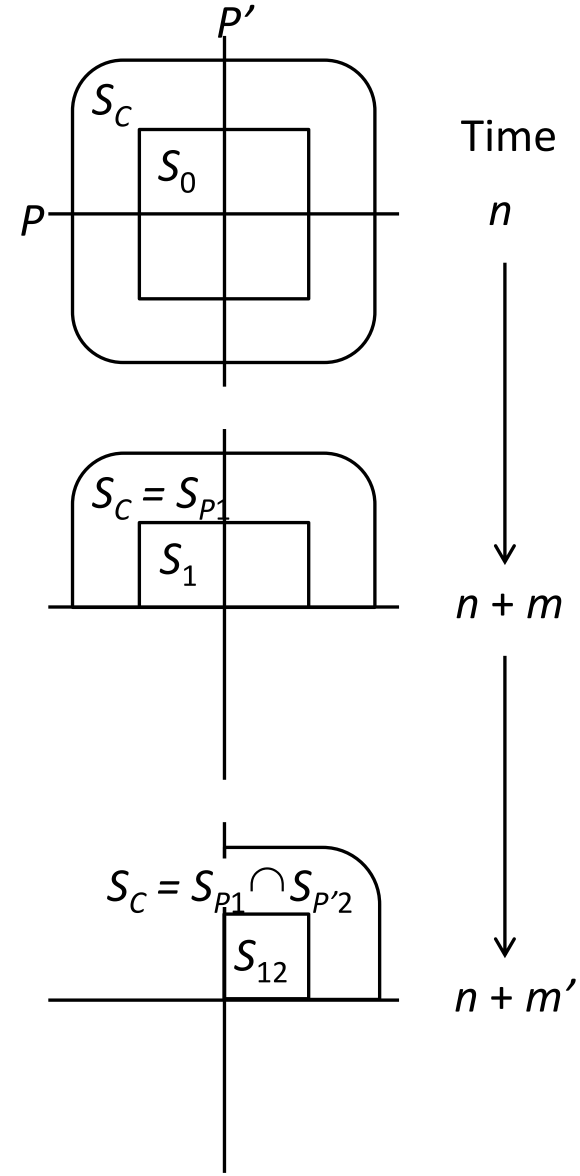
$$\overline{S}_C(n) = a_1\overline{S}_{P1} + a_2\overline{S}_{P2}$$

$$\overline{u}_C\overline{S}_C(n) = a_1\overline{S}_{P1} + a_2\overline{S}_{P2}$$

$$\overline{S}_C(n+m) = \overline{S}_{P1}$$

$$\overline{u}_C\overline{S}_C(n+m) = \overline{S}_{P1} = a_{11}\overline{S}_{PP'11} + a_{12}\overline{S}_{PP'12}$$

$$\overline{S}_C(n+m') = \overline{S}_{PP'12}$$



Algebraic representation of Mach-Zehnder contexts

P and P' at knowability levels 1 and 3, respectively (case c)

$$\overline{S}_C(n) = a_1\overline{S}_{P1} + a_2\overline{S}_{P2}$$

$$\overline{u}_C\overline{S}_C(n) = a_1\overline{S}_{P1} + a_2\overline{S}_{P2}$$

Hypothetical contextual state if the property value p_2 were known

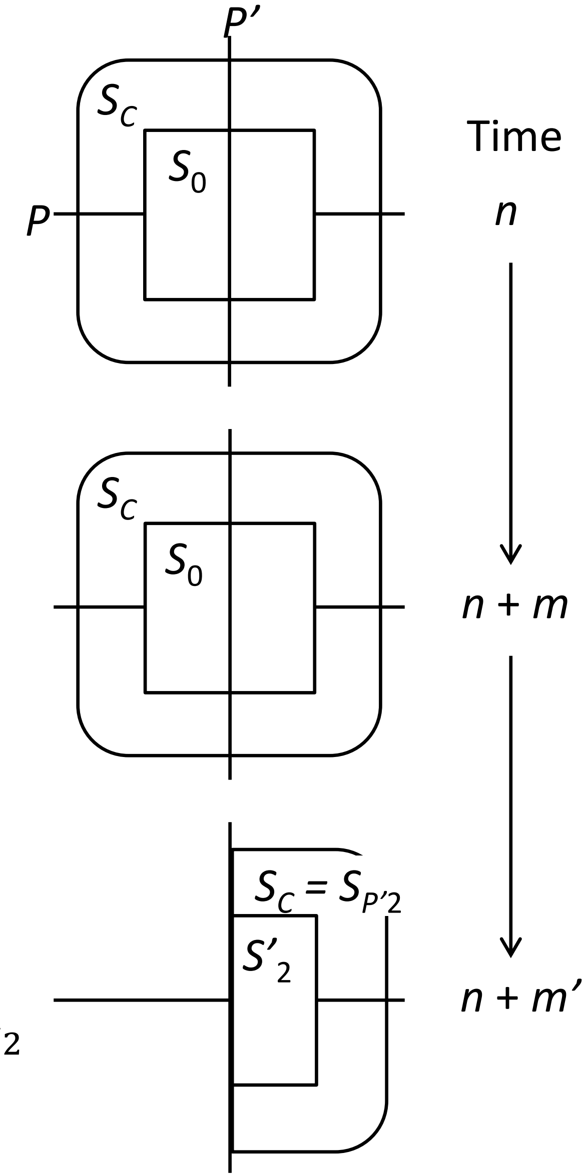
$$\overline{S}_C(n+m) = a_1\overline{S}_{C1}(n+m) + a_2\overline{S}_{C2}(n+m)$$

$$\overline{S}_{Ci}(n+m) = \overline{S}_{Pi}$$

$$\overline{u}_C\overline{S}_{Ci}(n+m) = a_{i1}\overline{S}_{P'1} + a_{i2}\overline{S}_{P'2}$$

$$\overline{u}_C\overline{S}_C(n+m) = a_1\overline{u}_C\overline{S}_{C1}(n+m) + a_2\overline{u}_C\overline{S}_{C2}(n+m)$$
$$= (a_1a_{11} + a_2a_{21})\overline{S}_{P'1} + (a_1a_{12} + a_2a_{22})\overline{S}_{P'2}$$

$$\overline{S}_C(n+m') = \overline{S}_{P'2}$$



Conditions on $f(a)$ in the algebraic representation of S_C

1) The presence of the two alternatives associated with P in context c) should be reflected in the representation, like in the previous slide, since the knowledge that the photon must pass one of the two mirrors is part of the context (epistemic completeness).

However, the representation should not be the same in context c) as in b), where we know which alternative is realized (explicit epistemic minimalism). In b) we get

$$q(p'_j) = q(p_1)q(p'_j|p_1)+q(p2)q(p'_j|p_2) = v_1\, v_{j1}+v_2\, v_{j2} = f(a_1)f(a_{1j}) + f(a_2)f(a_{2j})$$

In c) we get

$$q(p'_j) = v'_j = f(a_1 a_{1j}+ a_2 a_{2j})$$

The expressions are different, as required, if and only if

$$f(a) \neq a$$

Conditions on $f(a)$ in the algebraic representation of S_C

2) The relative volumes of any partition of a state add to one. Therefore

$$
\begin{aligned}
1 &= f(a_1) + f(a_2) \\
1 &= f(a_{11}) + f(a_{12}) \\
1 &= f(a_{21}) + f(a_{22}) \\
1 &= f(a_1 a_{11} + a_2 a_{21}) + f(a_1 a_{12} + a_2 a_{22})
\end{aligned}
$$

3) In context b) we may see (P, P') as *one* combined property with four possible values (p_1, p_1'), (p_1, p_2'), (p_2, p_1'), (p_2, p_2'), finally determined at time $n + m'$. Then we should write

$$\overline{u}_C \overline{S}_C(n+m) = a_1 a_{11} \overline{S}_{PP'11} + a_1 a_{12} \overline{S}_{PP'12} + a_2\, a_{21} \overline{S}_{PP'21} + a_2 a_{22} \overline{S}_{PP'22}$$

so that

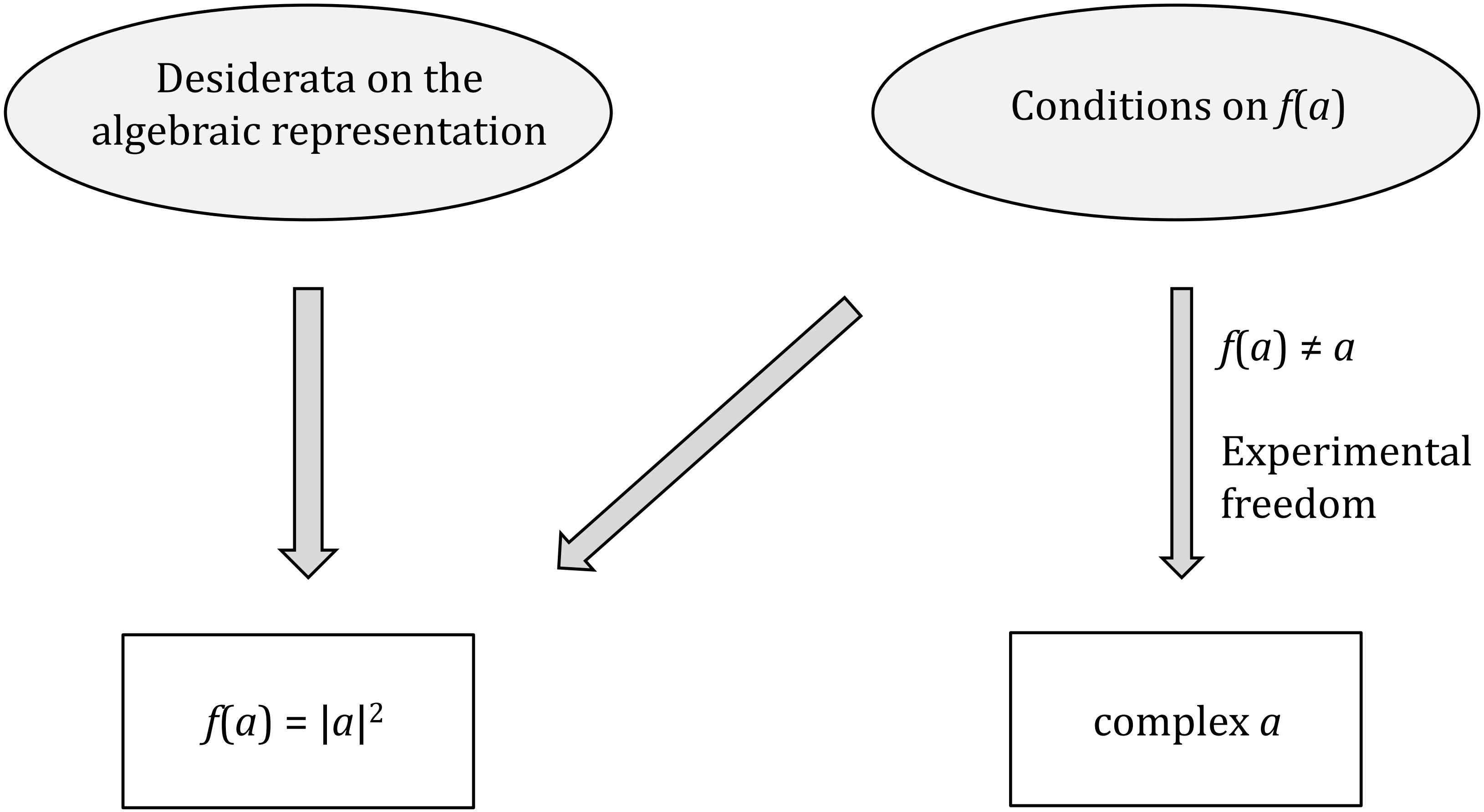
$f(a_i a_{ij}) = f(a_i) f(a_{ij})$.

Conditions on $f(a)$ in the algebraic representation of S_C

4) *Property independence.* The set of numbers $\{a_j\}$ reflects the arrangement of the part of the observational context that makes it possible to observe P, and the set $\{a_{ij}\}$ reflects the part with which P' is observed. These parts can be arranged independently, so that it should be possible to choose $\{a_{ij}\}$ independently from $\{a_j\}$. The choice of $f(a)$ should make this possible.

5) *Experimental freedom*. If there are M possible values for P and N possible values for P', then there are $M - 1$ independent relative volumes v_i and $M(N - 1)$ independent relative volumes v_{ij}, given the normalisations $1 = \sum_i v_i$ and $1 = \sum_i v_{ij}$. These degrees of freedom reflect a freedom to choose experimental setup and there should therefore be at least equally many independent numbers a_i and a_{ij} if the representation is to fulfil Desiderata 4). The choice of $f(a)$ should make this possible.

Born's rule



See the manuscript *A strict epistemic approach to physics* for details

Inner products

$$
\begin{aligned}
1 &= f(a_1 a_{11} + a_2 a_{21}) + f(a_1 a_{12} + a_2 a_{22}) \\
1 &= |a_1 a_{11} + a_2 a_{21}|^2 + |a_1 a_{12} + a_2 a_{22}|^2
\end{aligned}
$$

$$0 = a_1 a_2^{*}\left(a_{11}^{*} a_{21} + a_{12}^{*} a_{22}\right)$$

Property independence

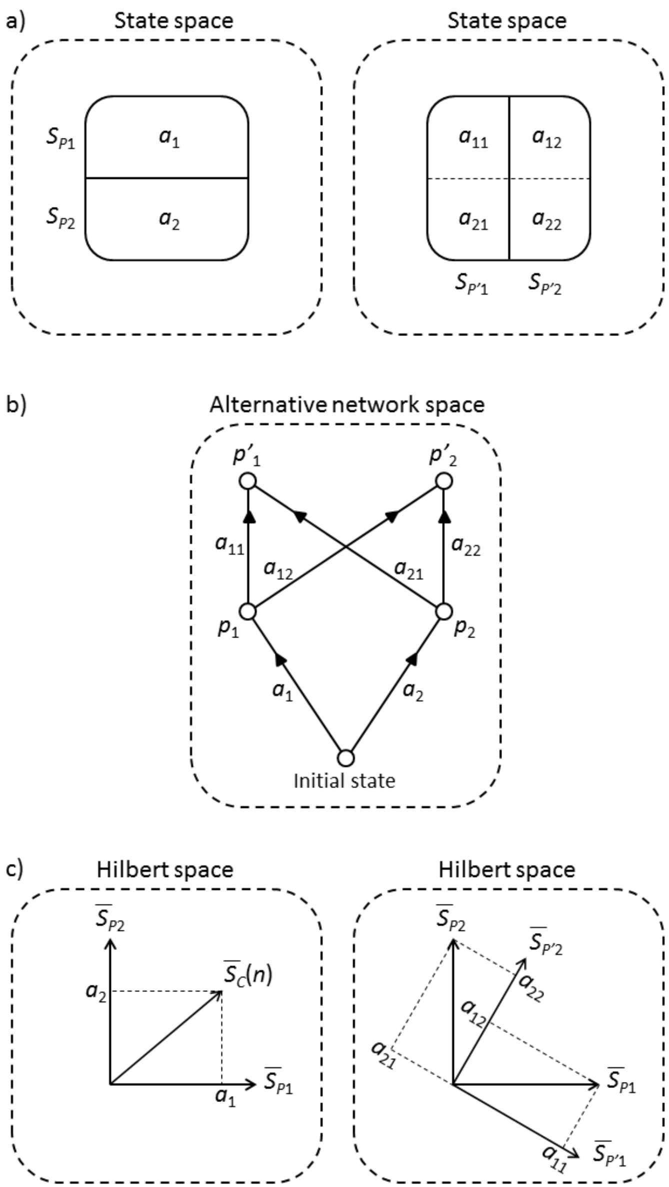
$$0 = a_{11}^{*} a_{21} + a_{12}^{*} a_{22}$$

Define formally: $\delta_{ij} = \langle \overline{S}_{Pi}, \overline{S}_{Pj} \rangle$

$$
\begin{aligned}
0 &= \langle a_{11}\overline{S}_{P'1} + a_{12}\overline{S}_{P'2}, a_{21}\overline{S}_{P'1} + a_{22}\overline{S}_{P'2} \rangle \\
&= \langle \overline{u}_C \overline{S}_{C1}, \overline{u}_C \overline{S}_{C2} \rangle
\end{aligned}
$$

Generally: $\delta_{ij} = \langle \overline{u}_C \overline{S}_{Ci}, \overline{u}_C \overline{S}_{Cj} \rangle$

Three representations of the observational context C



$$\overline{S}_{Ci}(n+m) = \overline{S}_{Pi}$$
$$\overline{u}_C\overline{S}_{Ci}(n+m) = a_{i1}\overline{S}_{P'1} + a_{i2}\overline{S}_{P'2}$$

$$\overline{S}_{Pi} = a_{i1}\overline{S}_{P'1} + a_{i2}\overline{S}_{P'2}$$

$$a_{ij} = \langle \overline{S}_{Pi}, \overline{S}_{P\prime j} \rangle$$

The Hilbert space representation is possible thanks to the possibility to define inner products and ortho-normality relations, as shown above.

Born's rule can then be expressed as

$$q(p_j{}') = \langle \overline{u}_C\overline{S}_C(n+m), \overline{S}_{P\prime j} \rangle$$

Hilbert space representations in different kinds of contexts

- The Hilbert space representation was seen to be almost unavoidable in the case where two properties *P* and *P*' are observed in sequence, and the values of *P* are unknowable (knowability level 1).

Let us consider two other cases:

- *P* and *P*' are simultaneously knowable, and both have knowability level 3.
- *P* and *P*' are *not* simultaneously knowable, and both have knowability level 3.

Let us see whether the same notions of orthonormal bases of property value states, of projections and Born's rule, can be applied in these cases also, so that the Hilbert space representation becomes general valid in all observational contexts C.

Principles to uphold in Hilbert space representations of observational contexts

Mutually exclusive states are represented by orthonormal subspaces in $\mathcal{H}_C$.

Examples: Two property values states S_{Pj} and $S_{Pj'}$, and two hypothetical contextual states S_{Ci} and $S_{Ci'}$ (applying just after the distinct but unknowable events that the value of P at knowability level 1 turns out to be p_j and $p_{j'}$).

The dimension D_H of $\mathcal{H}_C$ is chosen epistemically as follows:

D_H = the maximum number of independent property values known at the same time during the observational context C.

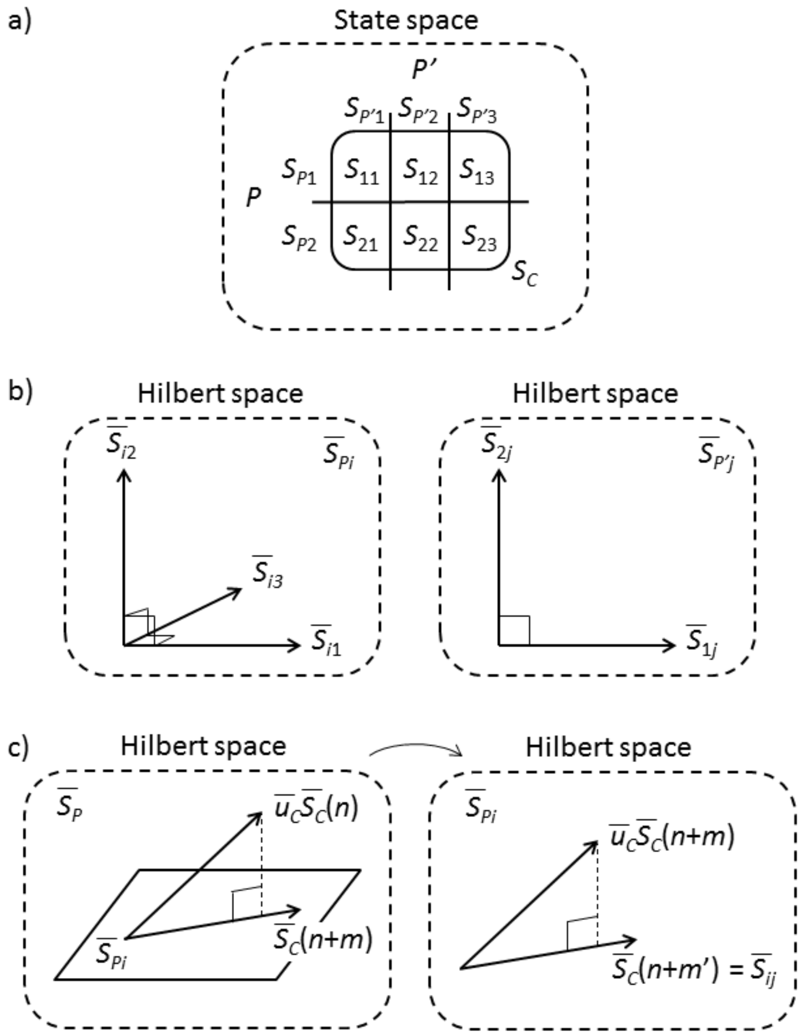
Example: In the Mach-Zehnder context in which P and P' are observed and P has knowablity level 1, we get $D_H = 2$ = the number of possible values of P'

Hilbert space representation when P and P' are simultaneously knowable

Suppose that P and P' have M and M' possible values, respectively.

When P has knowability level 1. We got $D_H = M'$.

Here P has knowability level 3. We get $D_H = MM'$.



In this example,
$M = 2$ and $M' = 3$,
so that $D_H = 6$.

The property value spaces S_{Pi} of P and $S_{P'j}$ of P' get dimensions 3 and 2, respectively

Hilbert space representation when P and P' are not simultaneously knowable

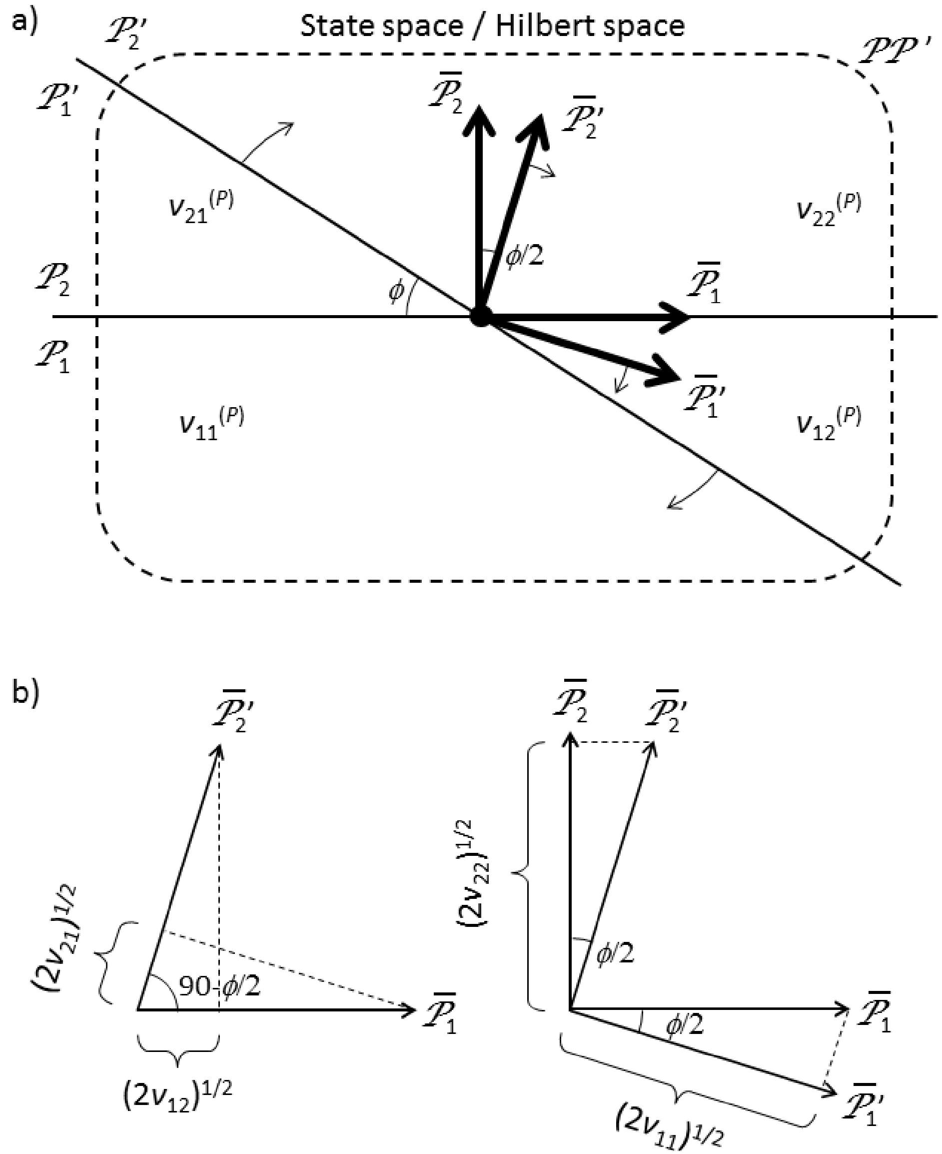
No problem to represent the combined property space $\mathcal{PP}'$ as a Hilbert space $\mathcal{H}_{PP'}$, particularly when P and P' has just two allowed property values each ($M = M' = 2$), like the spin s_x of an electron.

Two values knowable at the same time implies $D_H = 2$.

The definition of state space measure V implies $v_{12} = v_{21}$ and $v_{11} = v_{22}$.

To conform formally with Born's rule, we require

$$\delta_{ij} = \langle \overline{P}_i, \overline{P}_j \rangle$$
$$\delta_{ij} = \langle \overline{P}_i', \overline{P}_j' \rangle$$
$$\sqrt{2 v_{ij}{}^{(P)}} e^{i\theta_{ij}} = \langle \overline{P}_i, \overline{P}_j' \rangle$$



Hilbert space representation when P and P' are not simultaneously knowable

The above construction is abstract. It does not relate to any actual observational context C; There is no contextual state vector $\bar{S}_C$ defined in $\boldsymbol{\mathcal{H}}_{PP'}$.

The abstract construction can be applied to C when it is possible to choose bases $(\bar{S}_{P1}, \bar{S}_{P2})$ and $(\bar{S}_{P1}{}', \bar{S}_{P2}{}')$ with the same mutual relation as that between $(\bar{P}_1, \bar{P}_2)$ and $(\bar{P}_1{}', \bar{P}_2{}')$.

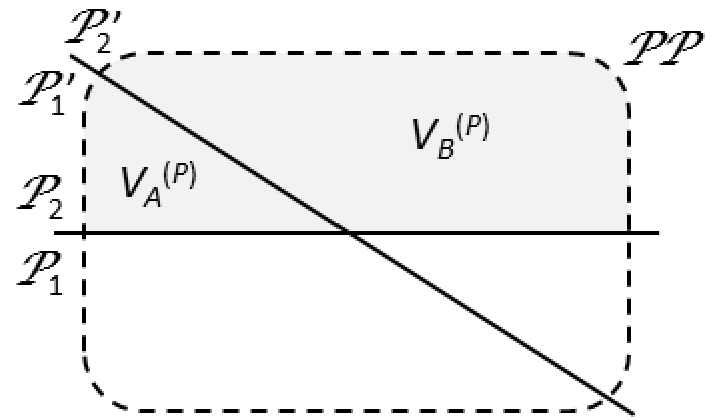
This is possible in **neutral contexts** C, which does not color this relation, where

$V_A/V_B = V_A{}^{(P)}/V_B{}^{(P)}$

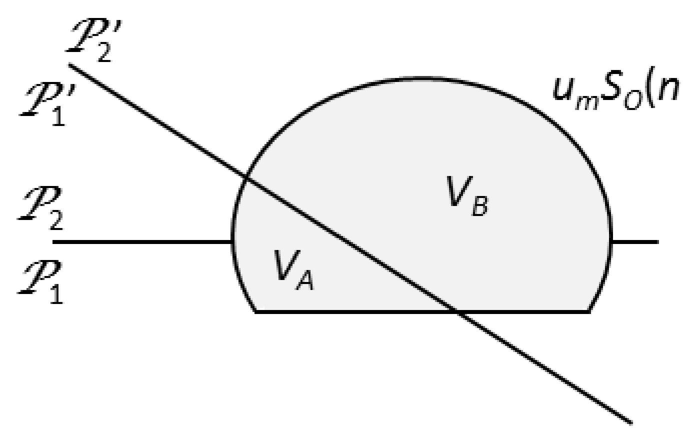
or

$$v_{ij} = \left|\langle \bar{P}_i, \bar{P}_j{}' \rangle\right|^2$$

a)



b)



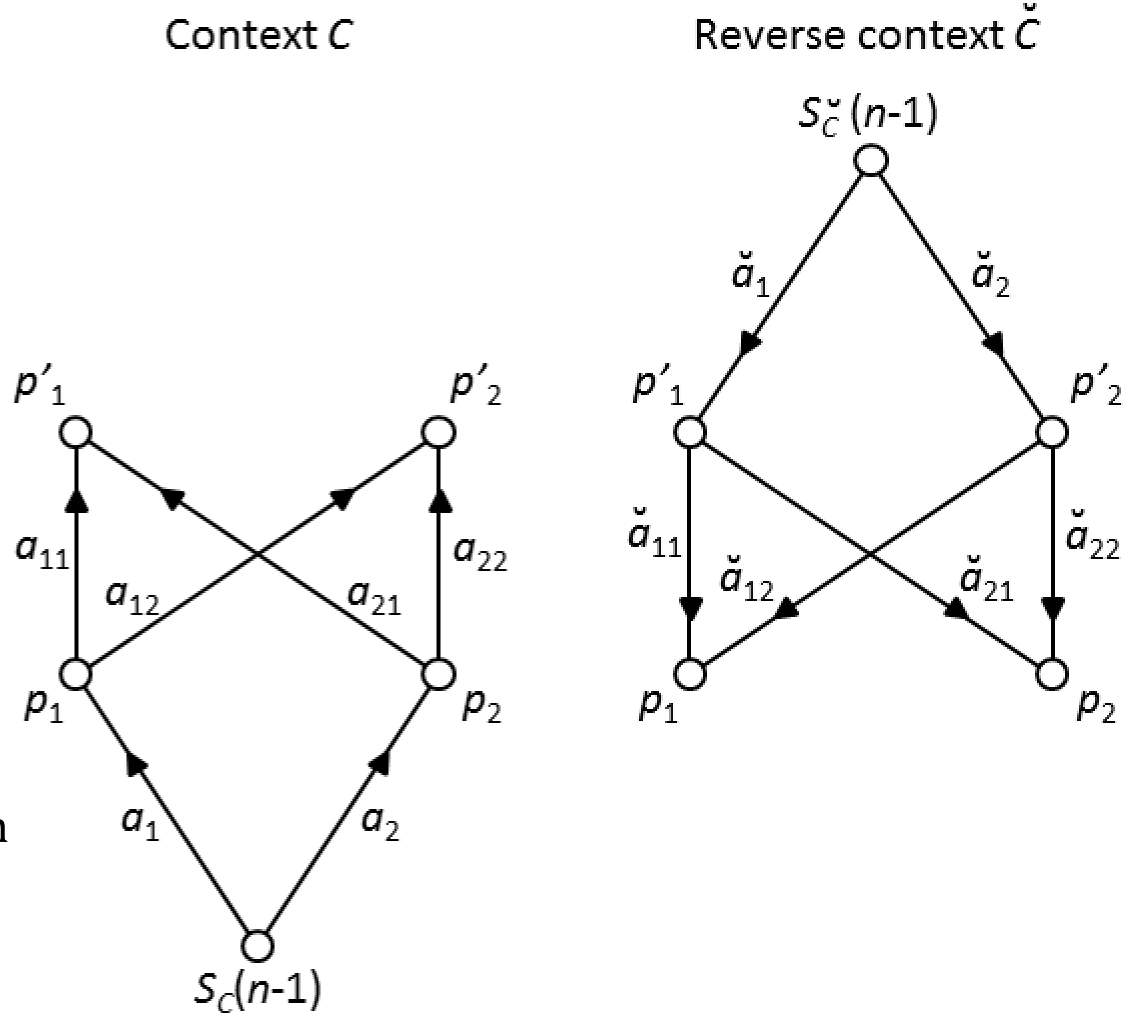
Hilbert space representation when *P* and *P*' are not simultaneously knowable

In that case we may formally set

$\mathcal{H}_C = \mathcal{H}_{PP'}$
$\bar{S}_{Pi} = \bar{P}_i$
$\bar{S}_{P'j} = \bar{P}_j{}'$

However, the two bases $(\bar{S}_{P1}, \bar{S}_{P2})$ and $(\bar{S}_{P'1}, \bar{S}_{P'2})$ associated with *P* and *P*' are not on equal footing in H_C since *P* is observed first, then *P*'.

We can symmetrize the situation if we consider a reverse context $\breve{C}$ in conjunction with *C*.



Change of basis in Hilbert space

To the original context C is associated a Hilbert space H_C with a contextual state vector $\bar{S}_C$.

To the reciprocal context $\breve{C}$ is associated a reciprocal Hilbert space $H_{\breve{C}}$ with a reverse contextual state vector $\bar{S}_{\breve{C}}$.

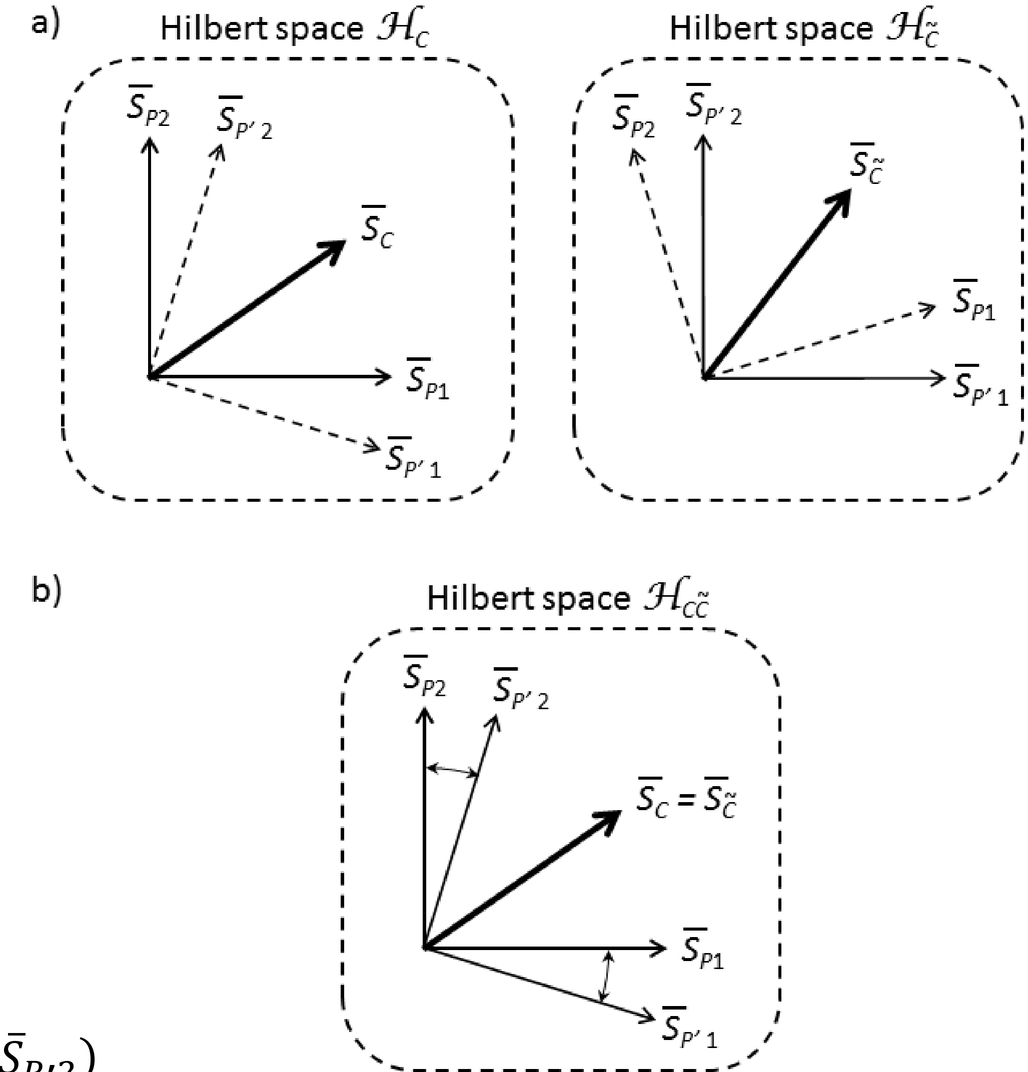
If $\breve{a}_i = a_1 a_{1i} + a_2 a_{2i}$ and $\breve{A} = A^{-1}$ with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then we call $\breve{C}$ the reverse context to C and denote it $\tilde{C}$.

In that case we can consider C and $\tilde{C}$ together in a combined Hilbert space $H_{C\tilde{C}}$ and identify $\bar{S}_C = \bar{S}_{\tilde{C}}$.

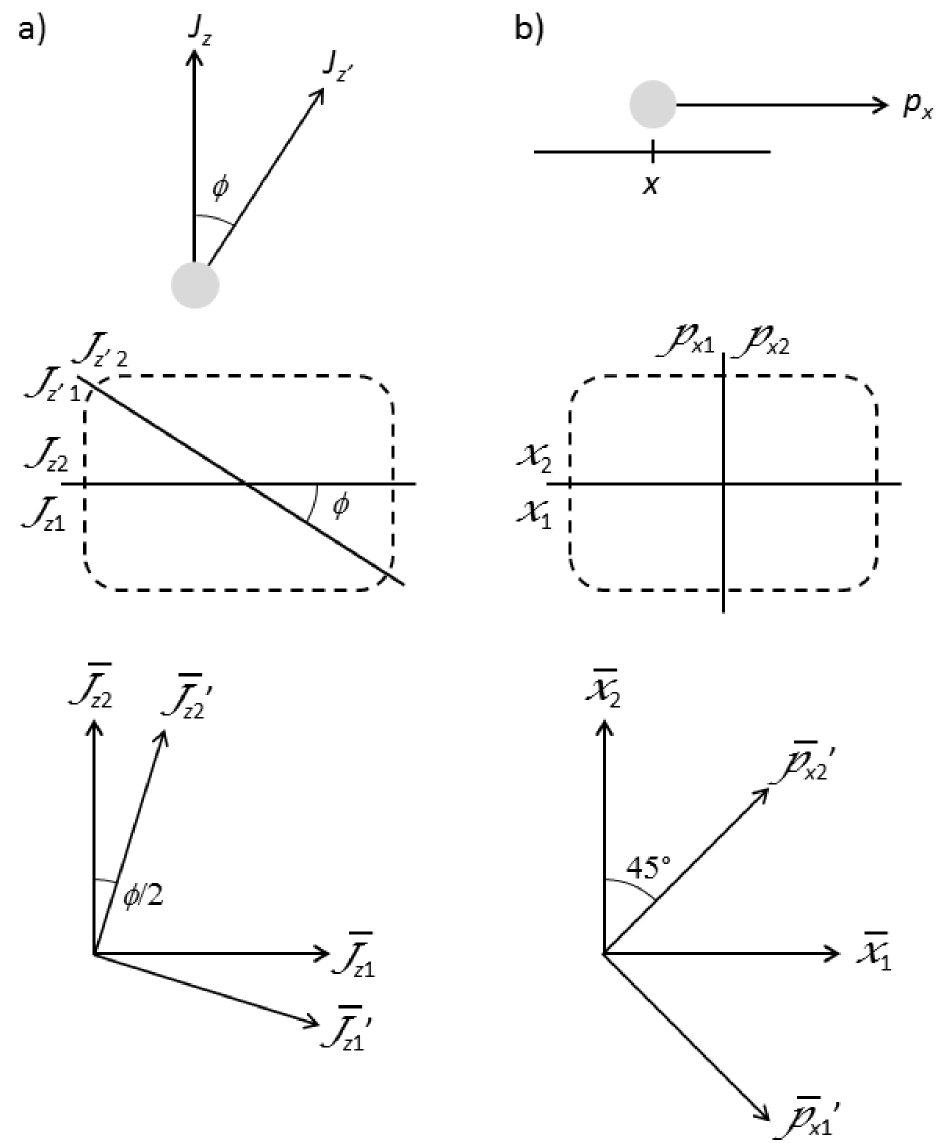
Then the two bases $(\bar{S}_{P1}, \bar{S}_{P2})$ and $(\bar{S}_{P\prime 1}, \bar{S}_{P\prime 2})$ are finally on equal footing and we can change bases in $H_{C\tilde{C}}$ as usual.



Examples of Hilbert space representation when *P* and *P*' are not simultaneously knowable

a) Mutually defined property pairs, like the angular momentum along two directions *z* and *z*' with an angle ϕ between them. We have $v_{ij} = v_{ji}$ for all *ij*.

b) Independent property pairs without any inherent relation that makes it more probable to observe a particular value of *P*' given the value of *P*. An example is position *x* and momentum p_x. We have $v_{ij} = v_{kl}$ for all *ij* and *kl*.



Hilbert space context representations – a summary

Consider the set *SC* of observational contexts *C* with a given specimen *OS*, a given sequence P, P', … of observed properties with given knowability levels, and given sets of possible values $\{p_j\}$, $\{p_j'\}$, … of these proeprties. To most such sets *SC* we can associate a complex vector space $\mathcal{H}_C$.

- Contexts with a pair of properties (P, P') where P has knowability level 1 And P' has knowability level 3 more or less forced a complex vector space representation upon us, where the probabilities to observe different values of P are given by Born’s law.

- Such a representation was seen to be possible also when both P and P' has knowability level 3, regardless whether they are simultaneously knowable or not.

- Trivially, the same kind of representation is possible also if we consider contexts with just one observed property P, and contexts with more than two observed properties, by combining simpler contexts.

Properties and operators

We considered the formal Hilbert space $\mathcal{H}_{PP'}$ above, discussing observational contexts C with two properties P and P' which are not simultaneously knowable.

In general, to each property P we may associate a Hilbert space H_P.
The property value spaces $\mathcal{P}_j$ correspond to vectors $\bar{P}_j$ which span H_P and are such that $\delta_{ij} = \langle \overline{P}_i, \overline{P}_j \rangle$.

In this language, we may write

$$P \leftrightarrow \begin{cases} H_P & \\ \{\bar{P}_j\} & a\ complete\ basis\ for\ H_P \\ \{p_j\} & the\ set\ of\ possible\ values\ of\ P \end{cases}$$

This means that we can associate each property P with exactly one linear, self-adjoint operator $\bar{P}$ with domain H_P, with complete basis of eigenvectors $\{\bar{P}_j\}$, and with corresponding real eigenvalues $\{p_j\}$.

Properties and operators

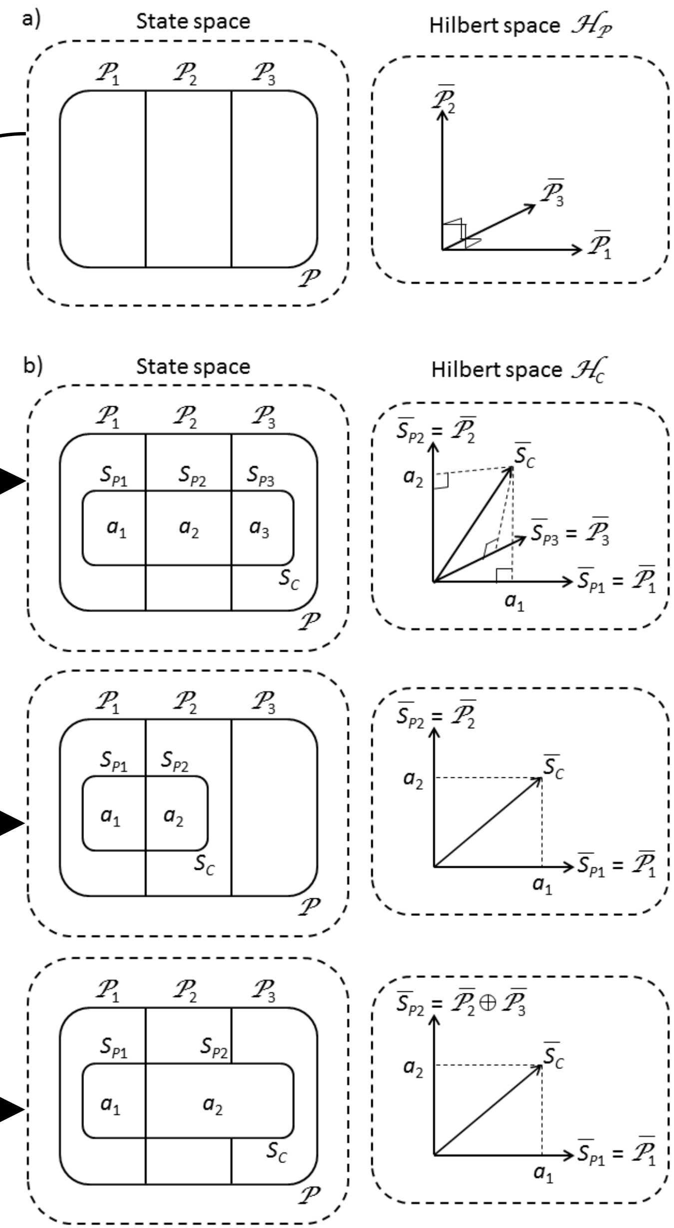
Above we related the abstract space $H_{PP'}$ with the space H_C that describes a concrete observational context C by the formal identification $\bar{S}_{Pi} = \bar{P}_i$.

Here, we can transfer the domain of $\bar{P}$ from H_P to H_C by saying $S_C \subseteq \mathrm{P}_i \Rightarrow \bar{P}\bar{S}_C = p_i \bar{S}_C$.
The linearity of $\bar{P}$ is then expressed by the rule

$$\begin{aligned} \bar{P}\bar{S}_C &= \bar{P}(a_1\bar{S}_{P1} + a_2\bar{S}_{P2} + a_3\bar{S}_{P3}) \\ &= a_1\bar{P}\bar{S}_{P1} + a_2\bar{P}\bar{S}_{P2} + a_3\bar{P}\bar{S}_{P3} \\ &= a_1p_1\bar{S}_{P1} + a_2p_2\bar{S}_{P2} + a_3p_3\bar{S}_{P3} \end{aligned}$$

If some value p_3 cannot be observed within context, the dimension of H_C decreases from 3 to 2, and $\bar{P}\bar{S}_C = a_1p_1\bar{S}_{P1} + a_2p_2\bar{S}_{P2}$

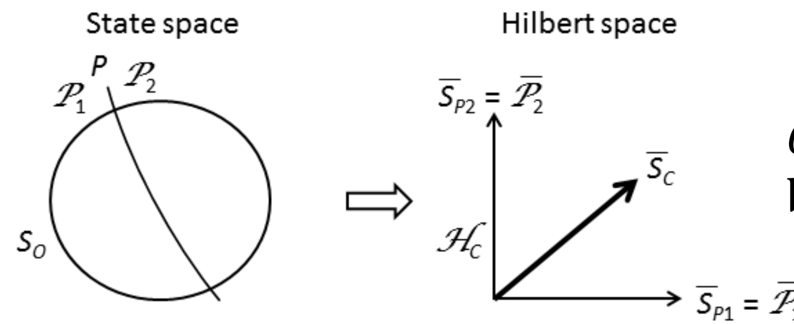
If the values p_2 and p_3 cannot be resolved, we define contextual property values $p_{C1} = p_1, p_{C2} = \{p_1, p_2\}$ and write $\bar{P}_C\bar{S}_C = a_1p_{C1}\bar{S}_{P1} + a_2p_{C2}\bar{S}_{P2}$ for the operator P_C corresponding to contextual property P_C.



Properties and operators

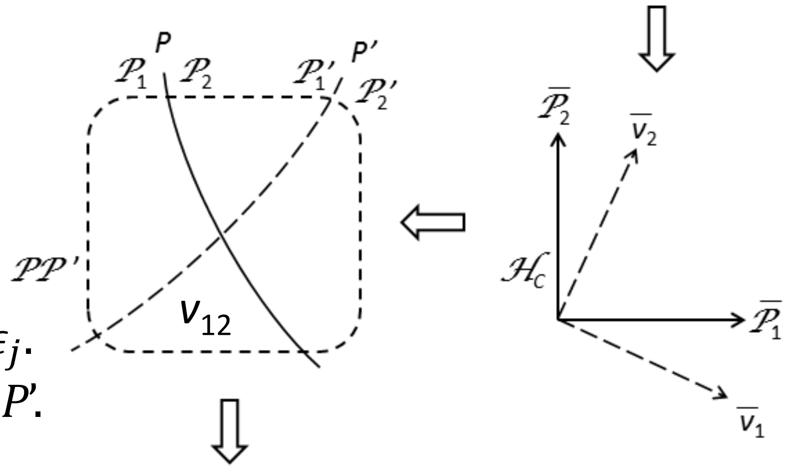
We have seen that to each property P we can associate exactly one self-adjoint operator $\bar{P}$. Conversely, to each such operator we can associate a property in the following sense.

Consider a context C
In which P is observed



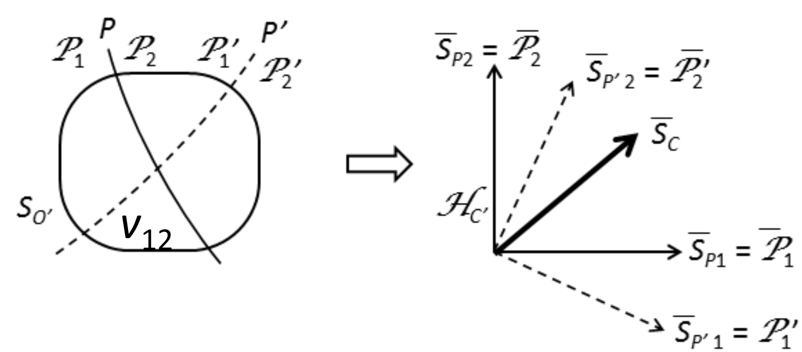
C can be represented by a Hilbert space H_C.

We can partition state space into disjoint sub-spaces $\mathscr{P}_j'$ so that $v_{ij} = \left|\langle \bar{P}_i, \bar{v}_j{}' \rangle\right|^2$. To $\mathscr{P}_j'$ we can associate value ϵ_j. This defines a property P'.



In such a Hilbert space we can introduce another basis $\{\bar{v}_j\}$. Choose real numbers $\{\epsilon_j\}$. We have then defined a self-adjoint operator $\bar{P}'$.

We can define a neutral context C' where P' is observed after P.



In H_C', $\{\bar{v}_j\}$ and $\{\epsilon_j\}$ are the eigenvectors and –values of the operator corresponding to P'.

Commutation rules for property operators

We have $[\bar{P}, \bar{P}'] \equiv 0$ if and only if properties P and P' are simultaneously knowable.

This is a consequence of the epistemic rule that the dimension D_H of the Hilbert space H_C is the maximum number of independent property values known at the same time during the observational context C.

Suppose that P and P' are not simultaneously knowable and have M possible values each. Then $D_H = M$.

If we were allowed to choose $D_H = M^2$ in this case, then we would have been able to deduce $[\bar{P}, \bar{P}'] \equiv 0$ if we let $\bar{P}$ and $\bar{P}'$ act on the basis vectors $\bar{S}_{ij}$ that correspond to the unattainable state of knowledge that P has value p_i and P' has value p_j'.