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Phase transitions towards frequency entrainment in large oscillator lattices

Per Östborn, Sven Åberg, and Gunnar Ohlén
Division of Mathematical Physics, Lund University, S-221 00 Lund, Sweden
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We simulate two-dimensional lattices of pulse-coupled oscillators with random natural frequencies, resembling pacemaker cells in the heart. As coupling increases, this system seems to undergo two phase transitions in the thermodynamic limit. First, the largest cluster of frequency entrained oscillators becomes macroscopic. Second, all oscillators frequency entrain, except possibly some isolated ones. Between the two transitions, the system has features indicating self-organized criticality. To our knowledge, the first transition and the intermediate phase have never been observed before. It remains to be seen if they are generic for large lattices of limit cycle oscillators.

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Many systems in science, engineering, and social life can be described as large networks of coupled limit cycle oscillators [1]. Most often there is a spread in the individual, natural frequencies, and the coupling is such that it tends to even out these frequency differences. A general question is how the dynamics of such systems changes when the coupling between the oscillators increases. Is there a phase transition at which the oscillators attain a common, collective frequency, in the thermodynamic limit where the number N of oscillators goes to infinity?

Such phase transitions are relevant in several fields. For example, the proper function of the 10^6 of pacemaker cells in the sinus node in the heart requires that they work at the same frequency. Cardiac arrhythmias may result if this is not the case [2]. The appearance of several frequencies in the sinus node may be caused by decoupling due to tissue degeneration. The brain also contains many pacemaker cells. Increasing evidence suggests that enhanced electric coupling between neighbor neurons can provoke epileptic seizures [3]. These correspond to pathologically large regions of synchronized electric discharges.

Most theoretical studies of phase transitions to collective oscillations assume that each oscillator is coupled to all the others, equally strong. The most well-known system of this kind is the Kuramoto model [4]. More realistic networks have local coupling. Winfree [5] and Kuramoto [6] hypothesized that in systems with nearest neighbor coupling and random natural frequencies, there should be a critical coupling strength at which the number of members, or size S_{max}, of the largest cluster of frequency entrained [7] oscillators becomes macroscopic. This can be expressed as a phase transition at which the order parameter r becomes nonzero, where

$$r \equiv \lim_{N\to\infty} S_{max}/N. \tag{1}$$

For a long time, model studies only revealed negative or inconclusive results regarding the existence of such a phase transition [8]. Recently, we proved that it is present in a one-dimensional chain of pulse-coupled oscillators where the natural frequencies have finite bandwidth [9]. A critical coupling strength g_c was found, at which global frequency entrainment settles and r jumps discontinuously from zero to one. In this paper, a two-dimensional square lattice with bidirectional nearest neighbor coupling is studied. Two phase transitions at the critical couplings g_{c1} and g_{c2} are found. The order parameter r seems to become nonzero at g_{c1}. At g_{c2}, r becomes very close to one, when all oscillators entrain, except possibly some isolated ones. Note that we can only have one phase transition in a one-dimensional lattice. States with $0<r<1$ are impossible, since then a nonzero density of oscillators in the chain are not entrained with the presumed infinite cluster, and these necessarily cuts the infinite cluster into finite parts, so that we are left with a state where $r=0$.

In our model (cf. Ref. [9]), the state of oscillator k is given by the phase $\phi_k \in [0,1)$. The time evolution of the phase is given by

$$\dot{\phi}_k = 1/P_k + g h(\phi_k) \sum_{l\in n_k} \delta(\phi_l), \tag{2}$$

where P_k is the natural period of oscillator k and n_k is the set of its nearest neighbors. An oscillator l is said to fire when $\phi_l = 1$. Then, $\phi_l \to 0$ and a pulse is delivered to the neighbor k, so that its phase immediately shifts according to $\phi_k \to \phi_k + g h(\phi_k)$. This kind of system can model oscillators that interact with short pulses and are strongly attracted to their limit cycles. The function $g h(\phi_k)$ is called the phase response curve (PRC), where g is the coupling strength. Inspired by experiments on pacemaker cells in the heart [10], we assume the form of the PRC given in Fig. 1. This coupling tends to even out phase and frequency differences be-

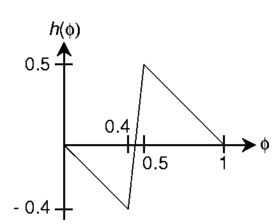

FIG. 1. The shown function $h(\phi)$ times the coupling constant g is the PRC used in system (2). From the requirement $0 \leqslant \phi + g h(\phi) < 1$, we have $g<1$. As discussed in Ref. [9], $g=1$ corresponds to infinite coupling.

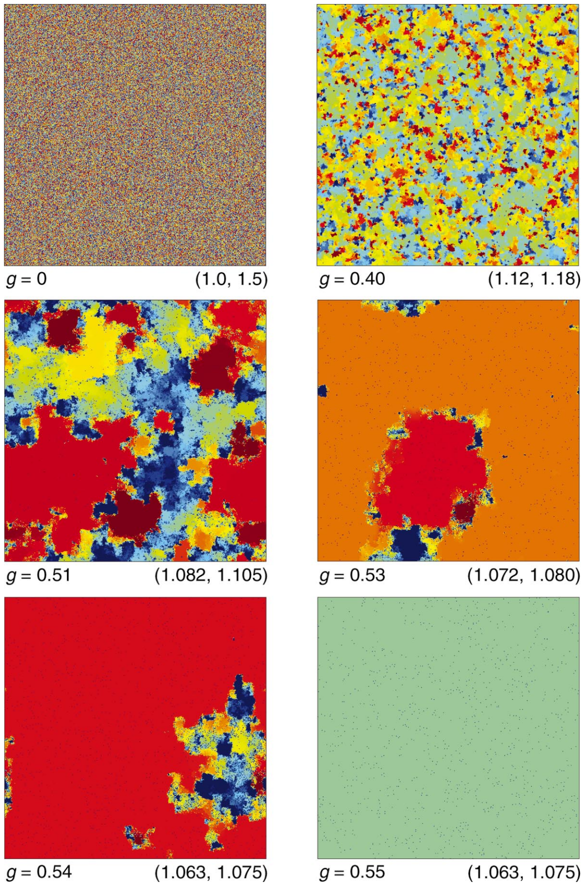

FIG. 2. (Color) Mean period landscapes in a lattice of 500 $\times$500 oscillators for different coupling strengths g. The couplings $g=0$ and $g=0.40$ belong to phase 1 where there are only microscopic frequency entrained clusters. At $g=0.51$, we are close to the critical value g_{c1}, where one cluster starts to dominate. The couplings $g=0.53$ and $g=0.54$ belong to phase 2, with one percolating, macroscopic cluster. At $g=0.55$, we have reached phase 3, where all oscillators frequency entrain, except for some silent oscillators (blue dots). The color codes for the mean period P in such a way that deep red corresponds to small $P\leqslant P_-$ and deep blue to large $P\geqslant P_+$. In each panel, the color scale is given as (P_-,P_+).

tween oscillators for the following reason: If the phase ϕ_k of an oscillator k receiving a pulse from a neighbor l is small, it becomes even smaller $(h<0)$, approaching $\phi_l=0$. If ϕ_k is large, it becomes even larger $(h>0)$, again approaching ϕ_l.

The natural periods P_k are taken as random numbers from a square distribution with $P_{min}=1$ and $P_{max}=1.5$ t.u. (time units). We use periodic boundary conditions. The same method of numerical integration as in Ref. [9] is used. The lattice is divided into blocks of 10×10 oscillators, within which the integration is exact. These correspond to the segments of 25 oscillators within which the integration was exact in Ref. [9].

The two phase transitions separate three phases. We shall say that we are in "phase 1" when $g<g_{c1}$, that we are in "phase 2" when $g_{c1}<g<g_{c2}$, and that we are in "phase 3" when $g>g_{c2}$. Figure 2 shows mean periods in a lattice of 500×500 oscillators measured during 10^4 t.u. after a transient of 10^5 t.u. for different coupling strengths g. For $g=0$, the mean periods are the natural periods, which are independent random numbers. As we increase g, clusters of oscillators with nearly identical mean periods appear, as can be seen for $g=0.40$. The typical size of these clusters increases with g. At $g=0.51$, one cluster is almost percolating horizontally, indicating that this coupling is close to the critical value g_{c1}. At $g=0.53$ and $g=0.54$, one cluster is percolating through the lattice. This cluster can be interpreted as macroscopic, suggesting that we have entered phase 2. For $g=0.55$, the entire lattice attains the same frequency, except for a few oscillators that never fire $(<0.05\%)$, suggesting that we have entered phase 3. The silent oscillators that never fire during the measurement interval start to appear around $g=0.48$.

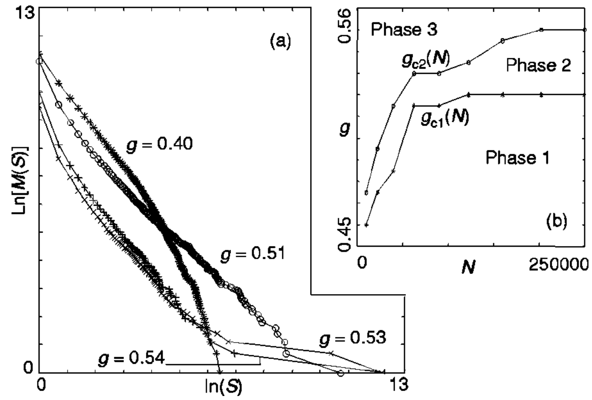

FIG. 3. (a) The number of clusters $M(S)$ with size equal to or larger than S as a function of S for different coupling strengths g in double logarithmic scale. The data are taken from the systems shown in Fig. 2. The distribution is subcritical for $g=0.4$, approximately critical for $g=0.51$, and close to critical also for $g=0.53$ and $g=0.54$, if the largest clusters are disregarded. (b) Estimations of $g_{c1}(N)$ and $g_{c2}(N)$ with accuracy $\Delta g=0.005$, using a single realization.

To locate g_{c1} more precisely, we study how the distribution $n(S)$ of cluster sizes S depends on g. A cluster is numerically defined to be a connected set of oscillators whose mean periods differ less than $dP=0.001$. The results are shown in Fig. 3(a). The distribution is sub-critical for low values of g, but the size of the largest cluster steadily increases with g. At $g=0.51$, the distribution becomes approximately critical, $n(S)\propto S^{-\tau}$, suggesting that g_{c1} is close to 0.51. The same evolution of $n(S)$ with g is seen in other realizations of the system, i.e., when the assignment of natural periods and the initial condition (phases at time zero) are changed. The uncertainty in g_{c1} is then seen to be ≈0.005. Subtracting one from the slope in the cumulative plots, we always get the critical exponent $\tau\approx2$, but the fluctuations are too large to determine a more exact value.

In phase 2, at $g=0.53$ and $g=0.54$, we interpret the cluster size distributions still to be critical. To do so, we have to exclude the largest clusters, which are too large to fit in, and the smallest, which are too many. The latter is not surprising, since scale invariance is never expected to hold on the smallest scales. For the larger clusters, we never see that the distribution bends downwards in the double logarithmic plots [cf. Fig. 3(a)], indicating subcriticality. In some realizations of phase 2, the size difference between the largest clusters that must be disregarded and the smaller ones is not so large as in Fig. 3(a). Then, the critical part of the distribution extends further up in size. (In fact, we do not always see a percolating cluster in phase 2, maybe due to finite time effects. However, the largest clusters are always larger than they would be if all cluster sizes were critically distributed.) From this evidence, we hypothesize that the entire phase 2 is critical. This is further supported by the fact that the features described above are robust with respect to the cluster discrimination parameter dP. Self-organized criticality has been observed previously in lattices of pulse-coupled oscillators with diverse natural frequencies [11], where it appeared as critically distributed avalanches of simultaneous firings. The question of frequency entrainment was not addressed in that study. In our model, such avalanches are, however, impossible, since waves of firings propagate with finite speed whenever $g<1$.

To confirm the existence of g_{c1} and g_{c2}, one should ideally determine their magnitude as a function of N, and see whether they converge to finite, separate values as $N\rightarrow\infty$. Figure 3(b) shows estimations of $g_{c1}(N)$ and $g_{c2}(N)$ with accuracy $\Delta g=0.005$, using a single realization for each N. More realizations indicate that the spread of the critical values is similar to Δg for $N=500^2$, i.e., considerably less than $g_{c2}(N)-g_{c1}(N)$. The shape of the curves in Fig. 3(b) support the hypothesis that the critical couplings converge to separate finite values. Since a critical coupling $g_c=\sqrt{2/3}\approx 0.82$ for global frequency entrainment was shown to exist in the corresponding one-dimensional lattice [9], we are strongly inclined to believe that a finite $g_{c2}<0.82$ exists in two dimensions, since increased connectivity facilitates the appearance of order. Therefore, the most important observation in Fig. 3(b) is that $g_{c2}(N)-g_{c1}(N)$ does not seem to decrease as N increases, suggesting that g_{c1} is indeed lower than g_{c2}, and that phase 2 exists in the thermodynamic limit $N\rightarrow\infty$. It should also be checked that a lattice in phase 2 remains in this phase and does not converge to phase 3 as $t\rightarrow\infty$. To do so, we simulate a lattice of 300×300 oscillators at $g=0.525$ for very long time $(4.4\times 10^5$ t.u.). After a transient of about 1.5×10^5 t.u., the standard deviation of the mean period distribution in the lattice does not show any tendency to decrease (not shown), so that the lattice indeed does not approach phase 3.

Mean period landscapes from this long simulation in phase 2 are shown in the bottom row of Fig. 4. The cluster configuration never seems to stabilize. This is consistent with the hypothesis that the entire phase 2 is critical, since then the system should have infinite transient time. The upper row of Fig. 4 shows mean periods in the same lattice for $g=0.40$ (in phase 1), calculated at corresponding times. It is seen that the positions of the clusters remain essentially fixed. This seems always to be the case in phase 1. A related instability in phase 2 is that given the assignment of natural periods, the mean period landscapes at a given time from two simulations with different initial conditions look very different. This would make the landscape in phase 2 sensitive to external perturbations. In contrast, in phase 1 the system seems to approach the same mean period landscape regardless of the initial condition. In phase 3, the position and exact number of silent oscillators depend on the initial condition. However, we stress that the *statistical* properties of the system seem to depend only on g in the thermodynamic limit, and not on the initial condition or the assignment of natural periods.

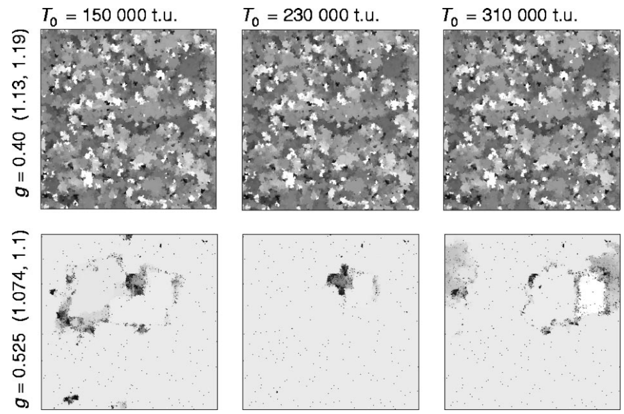

FIG. 4. Evolution of mean period landscapes in a lattice of 300×300 oscillators for $g=0.40$ (top row), belonging to phase 1, and $g=0.525$ (bottom row), belonging to phase 2. The clusters seem stable in phase 1 and unstable in phase 2. The mean periods were measured during 10^4 t.u., starting from different times T_0. The color scale is given in brackets in the same way as in Fig. 2, but here white corresponds to small P and black to large P.

The mean period of an oscillator is defined as the measurement time $\Delta T\rightarrow\infty$. The instability of the clusters in phase 2 indicates that these numbers, if they exist, could be very different from the mean periods that we measure during a finite time (10^4 t.u.). We can imagine three possibilities: (1) The mean periods do not exist, (2) they exist and are the same for all oscillators, and (3) they exist and are different. If frequency clusters appear and disappear at completely random places in the network as time passes, then alternative two could be true. If not, we could have alternative three. Then, one could ask how the mean period landscape would look like. There might no longer be a macroscopic, percolating cluster. If this is the case, or alternative (1) or (2) is true, one has to assume a finite ΔT to say that $0<r<1$ in phase 2.

Figure 5 shows distributions $n(P)$ of the mean periods P for different values of g. It is seen that the maximum shifts towards smaller values of P as g increases. For a one-dimensional chain, it was shown that the entrained period was always that of the fastest oscillator in the thermodynamic limit [9]. The same could very well be true here, in which case the spike of the distribution would be placed at $P=1$ for $g>g_{c2}$ (or maybe already for $g>g_{c1}$). Above $g=0.45$, the distribution becomes progressively asymmetric, with a wider and wider tail of long periods. As seen in Fig. 5(c), this tail approximately obeys a power law $n(P)\propto(P-P_0)^{-\chi}$ in phase 2, with $\chi\approx 2$. The appearance of oscillators with long mean periods parallels the appearance of completely silent oscillators, mentioned previously. It is seen that oscillators with very long mean periods most often fire with normal intervals, but also experience very long periods of silence, appearing as intermittent "bursts." Above g_{c2}, the tail in $n(P)$ disappears.

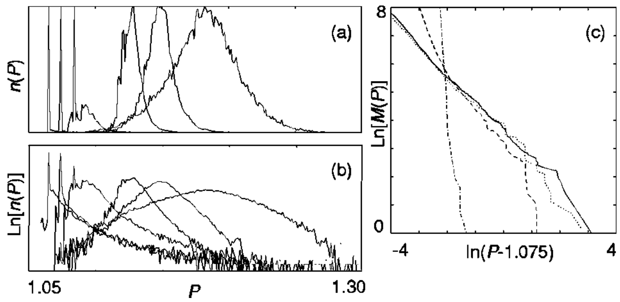

FIG. 5. Distributions $n(P)$ of mean periods P in a lattice of size 500×500. (a) Linear scale, normalized heights. The rightmost peak corresponds to $g=0.30$, and going to the left we have $g=0.40$, 0.45, 0.51, 0.53, and 0.54. (b) Corresponding distributions in logarithmic scale. (c) The long period tails in double logarithmic scale. $M(P)$ is the number of oscillators with mean period equal to or larger than P. The dash-dotted line corresponds to $g=0.40$, the dashed to $g=0.51$, the dotted to $g=0.53$, and the solid to $g=0.54$. For the two latter couplings in phase 2, the tails follow a power law.

In summary, we have found strong indications for the existence of two phase transitions in a large two-dimensional oscillator lattice with diverse natural frequencies. First, one frequency entrained cluster becomes macroscopic. Second, almost all oscillators frequency entrain. Between the two transitions, the system seems critical, with power law cluster size distribution $n(S)\propto S^{-\tau}$ for the microscopic clusters, and unstable cluster configuration. In this phase, there seems to be a large tail $(P-P_0)^{-\chi}$ of slow oscillators in the distribution $n(P)$ of mean periods P. The critical exponents τ and χ are both close to 2. However, we are not able to present a list of the relevant critical exponents for this system. We cannot say whether the order parameter r increases continuously from zero or not in phase 2, due to the large fluctuations of the size of the macroscopic cluster, both between different realizations, and as time passes. If it does, the first transition could be called second order. The fact that the tail of slow oscillators in $n(P)$ seems to develop continuously as we pass through g_{c1} support this hypothesis. The second phase transition seems to be first order in the sense that the cluster size distribution suddenly collapses at g_{c2}, just like the tail of slow oscillators. However, the transitions cannot be of traditional type if it is true that they are separated by a critical phase. This is a first account of these phenomena. Future studies should establish the properties of the transitions in more detail and investigate the conditions for their appearance with regard to the oscillator model and the distribution of natural periods.

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